OPTIMAL DECENTRALIZED LINEAR PRECODING FOR WIDEBAND NON-COOPERATIVE INTERFERENCE SYSTEMS BASED ON GAME THEORY

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ABSTRACT

In this paper we formulate the problem of finding the optimal precoding/multiplexing strategy in an infrastructureless multiuser scenario as a noncooperative game. We first consider the theoretical problem of maximizing mutual information on each link, given constraints on the spectral mask and transmit power. Then, to accommodate practical implementation aspects, we focus on the competitive maximization of the transmission rate on each link, using finite order constellations, under the same constraints as above plus a constraint on the average error probability. We prove that in both cases a NE always exists and the optimal precoding/multiplexing strategy leads to a (pure strategy) diagonal transmission for all the users. Thanks to this result, we can reduce both original complicated *matrix-valued* games to a simpler unified *vector* power control game. Thus, we derive sufficient conditions for the uniqueness of the NE of such a game, that are proved to have a broader validity than conditions known in the literature for special cases of our game. Finally, we show that the Nash equilibria of the vector game can be reached using the so-called asynchronous iterative waterfilling algorithm.

1. INTRODUCTION AND MOTIVATION

In this paper, we address the problem of finding the optimal precoding/multiplexing strategy for a multiuser system composed of a set of Q noncooperative wideband links, sharing the same physical resources, e.g., time and bandwidth. No multiplexing strategy is imposed a priori so that, in principle, each user interferes with each other. We consider block transmissions, as a general framework encompassing most current schemes like, e.g., CDMA or OFDM systems (it is also a capacity-lossless strategy for sufficiently large block length). Thus, each source transmits a coded vector

$$\mathbf{x}_q = \mathbf{F}_q \mathbf{s}_q, \tag{1}$$

where $\mathbf{s}_q$ is the $N\times 1$ information symbol vector and $\mathbf{F}_q$ is the $N\times N$ precoding matrix. Denoting with $\mathbf{H}_{rq}$ the channel matrix between source r and destination q, the sampled baseband block received by the q-th destination is (dropping the block index)

$$\mathbf{y}_q = \mathbf{H}_{qq}\mathbf{x}_q + \sum_{r\neq q=1}^{Q} \mathbf{H}_{rq}\mathbf{x}_r + \mathbf{w}_q, \tag{2}$$

where $\mathbf{w}_q$ is a zero-mean circularly symmetric complex Gaussian white noise vector with covariance matrix $\sigma_q^2\mathbf{I}$. The second term on the right-hand side of (2) represents the Multi-User Interference (MUI) received by the q-th destination and caused by the other active links. Treating MUI as additive noise, the estimated symbol vector at the q-th receiver is

$$\widehat{\mathbf{s}}_q = D\left[\mathbf{G}_q^H \mathbf{y}_q\right], \tag{3}$$

where $\mathbf{G}_q^H$ is the $N\times N$ receive matrix (linear equalizer) and $D[\cdot]$ denotes the decision operator that decides which symbol vector has been transmitted.

The system model above is sufficiently general to incorporate many cases of practical interest, such as: i) Digital subscriber lines, where the matrices $(\mathbf{F}_q)_{q=1}^Q$ incorporate DFT precoding and power allocation, whereas the MUI is mainly caused by near-end cross talk; ii) Cellular radio, where the matrices $(\mathbf{F}_q)_{q=1}^Q$ contain the user codes within a given cell, whereas the MUI is essentially intercell interference; iii) Ad hoc wireless networks, where there is no central unit assigning the coding/multiplexing strategy to the users. Moreover, system (2) is particularly appropriate for *cognitive radio* systems, where each user is allowed to re-use portions of the already assigned spectrum in an adaptive way, depending on the interference generated by other users.

Within this setup, the system design consists on finding the optimal matrix set $(\mathbf{F}_q, \mathbf{G}_q)_{q=1}^Q$ according to some performance measure and optimality criterion. Aiming at finding decentralized solutions with no coordination among the users, we adopt, as optimality criterion, the achievement of a Nash Equilibrium (NE) [1], and we focus on the following two strategic noncooperative (matrix-value) games: $\mathscr{G}_1$) competitive maximization of mutual information on each link, given constraints on transmit power and spectral radiation mask; and $\mathscr{G}_2$) competitive maximization of the transmission rate on each link, using finite order constellations, under the same constraints as above plus a constraint on the average (uncoded) error probability. The spectral mask constraints are useful to impose radiation limits over licensed bands, where it is possible to transmit but only with a spectral density below a specified value. Game $\mathscr{G}_2$ is motivated by the practical need of using discrete constellations, as opposed to Gaussian distributed symbols.

Because of the inherently competitive nature of a multi-user system, it is not surprising that game theory has been already adopted to solve many problems in communications. A vector power control game was proposed in [2] to maximize the information rates (under constraints on the transmit power) of two users in a DSL system, modelled as a frequency-selective Gaussian interference channel. The problem was extended to an arbitrary number of users in [3]-[5]. Vector power control problem in flat-fading Gaussian interference channels was addressed in [6].

The original contributions of this paper with respect to the current literature on vector games [2]-[6] are listed next. We consider two alternative *matrix-valued* games, whereas in [2]-[4], [6] the authors studied a *vector* power control game which can be obtained from $\mathscr{G}_1$ as special case, when the diagonal transmission is imposed a priori and no spectral mask constraints are considered. Conversely, we do not assume any a priori diagonalizing scheme and our study of game $\mathscr{G}_2$ is totally new. Our first contribution is to show that the solution set of both games is always nonempty and contains only deterministic strategies. More important, we prove that the diagonal transmission from each user through the channel eigenmodes (i.e., the frequency bins) is optimal, irrespective of the channel state, power budget, spectral mask constraints, and interference levels. This result yields a strong simplification of the original optimization, as it converts both complicated *matrix-valued* games $\mathscr{G}_1$ and $\mathscr{G}_2$ into a simpler unified *vector* power control game, with no performance penalty. Interestingly, such a simpler vector game includes, as special cases, the games studied in [2]-[4], [6]. The second important contribution of the paper is to provide sufficient conditions for the uniqueness of the NE of our vector power control game that have broader validity than those given in [2]-[4], [6] (without mask constraints) and, more recently, in [5] (including mask constraints).

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Finally, we show that the NEs of the vector game can be obtained in a totally asynchronous way (in the sense of [9]) using the asynchronous Iterative Waterfilling Algorithm (IWFA) that we recently proposed to solve the rate maximization game in frequency-selective interference channels [10].

2. SYSTEM MODEL AND PROBLEM FORMULATION

Given the I/O system in (2), we make the following assumptions: **A.1**) Aiming at finding distributed algorithms, we focus on transmission techniques where neither user coordination nor interference cancelation is allowed, so that MUI is treated as additive colored noise; **A.2**) Each channel is modeled as an FIR filter of maximum order L_h and it is assumed to change sufficiently slowly to be considered fixed during the whole transmission; **A.3**) To facilitate symbol recovery, a cyclic prefix of length $L \geq L_h$ is incorporated on each transmitted block $\mathbf{x}_q$ in (1), so that each matrix $\mathbf{H}_{rq}$ in (2) resulting after having discarded the guard interval at the receiver, is a Toeplitz circulant matrix. Thus $\mathbf{H}_{rq} = \mathbf{W}\mathbf{D}_{rq}\mathbf{W}^H$, with $\mathbf{W} \in \mathbb{C}^{N\times N}$ denoting the normalized IFFT matrix, i.e., $[\mathbf{W}]_{ij} \triangleq e^{j2\pi(i-1)(j-1)/N}/\sqrt{N}$ for $i, j = 1, \ldots, N$ and $\mathbf{D}_{rq}$ is a $N \times N$ diagonal matrix, where $[\mathbf{D}_{rq}]_{kk} \triangleq \bar{H}_{rq}(k)/\sqrt{d_{rq}^{\gamma}}$ is the frequency-response of the channel between source r and destination q including the path-loss d_{rq}^{γ} with exponent γ and normalized fading $\bar{H}_{rq}(k)$, with d_{rq} denoting the distance between the transmitter r and the receiver q; **A.4**) The channel from each source to its own destination is known to the intended receiver, but not to the other terminals; each receiver is also assumed to get an error-free estimate of the covariance matrix of the whole disturbance coming from other users. Based on this information, each destination computes the optimal precoding matrix for its own link and transmits it back to its transmitter through a low (error-free) bit rate feedback channel.

The physical constraints required by the applications are:
Co.1 Maximum transmit power for each transmitter, i.e.,

$$E\left\{\|\mathbf{x}_q\|_2^2\right\} = \frac{1}{N}\mathsf{Tr}\left(\mathbf{F}_q\mathbf{F}_q^H\right) \leq P_q, \tag{4}$$

where P_q is power in units of energy per transmitted symbol, and the symbols are assumed to be, w.l.o.g., zero-mean unit energy uncorrelated symbols, i.e., $E\left\{\mathbf{s}_q(n)\mathbf{s}_q^H(n)\right\} = \mathbf{I}$.
Co.2 Spectral mask constraint, i.e., $\forall k \in \{1, \ldots, N\}$

$$E\left\{\left|[\mathbf{W}^H\mathbf{F}_q\mathbf{s}_q]_k\right|^2\right\} = \left[\mathbf{W}^H\mathbf{F}_q\mathbf{F}_q^H\mathbf{W}\right]_{kk} \leq \overline{p}_q^{\max}(k), \tag{5}$$

where $\overline{p}_q^{\max}(k)$ represents the maximum power that is allowed to be allocated on the k-th frequency bin from the q-th user.[1] Constraints in (5) are imposed by radio spectrum regulations and attempt to limit the amounts of interference generated by each transmitter over some specified frequency bands.
Co.3 Maximum tolerable (uncoded) symbol error rate (SER) on each link, i.e.,

$$P_{e,q}(k) \triangleq \text{Prob}\{\hat{s}_q(k) \neq s_q(k)\} \leq P_{e,q}^{\star}, \quad \forall k \in \{1, \ldots, N\}, \tag{6}$$

where $\hat{s}_q(k)$ is the k-th entry of $\hat{\mathbf{s}}_q$ given in (3).

2.1. Competitive maximization of mutual information

In this section we focus on the fundamental (theoretic) limits of system (2), under **A.1**-**A.4**, and consider the competitive maximization of information rate of each link, given constraints **Co.1**, **Co.2**. It is straightforward to see that a (pure or mixed strategy) NE is obtained if each user transmits using Gaussian signaling, with a proper precoding $\mathbf{F}_q$. Hence, given **A.4**, the mutual information for the q-th user is

$$\mathrm{I}_q(\mathbf{F}_q, \mathbf{F}_{-q}) = \frac{1}{N}\log\left(\left|\mathbf{I} + \mathbf{F}_q^H\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}\mathbf{H}_{qq}\mathbf{F}_q\right|\right) \tag{7}$$

[1] Observe that if $(1/N)\sum_k \overline{p}_q^{\max}(k) \leq P_q$, we obtain the trivial solution $[\mathbf{W}^H\mathbf{F}_q\mathbf{F}_q^H\mathbf{W}]_{kk} = \overline{p}_q^{\max}(k), \forall k$.

where $\mathbf{R}_{-q} \triangleq \sigma_q^2\mathbf{I} + \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{F}_r\mathbf{F}_r^H\mathbf{H}_{rq}^H$ is the interference plus noise covariance matrix, observed by user q, and $\mathbf{F}_{-q} \triangleq (\mathbf{F}_r)_{r\neq q=1}^Q$ is the set of all the precoding matrices, except the q-th one. Observe that, for each link, we can always consider that the receiver is composed of an MMSE stage plus some other stage, since the MMSE is capacity-lossless. Thus, w.l.o.g., we assume in the following that

$$\mathbf{G}_q = \mathbf{R}_{-q}^{-1}\mathbf{H}_{qq}\mathbf{F}_q(\mathbf{I}+\mathbf{F}_q^H\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}\mathbf{H}\mathbf{F}_q)^{-1}, \quad \forall q \in \{1, \ldots, Q\}. \tag{8}$$

Hence, the strategy of each player amounts to finding the optimal precoding $\mathbf{F}_q$ that maximizes $\mathrm{I}_q(\mathbf{F}_q, \mathbf{F}_{-q})$ in (7), under constraints **Co.1** and **Co.2**. Stated in mathematical terms, we have the following strategic noncooperative game

$$(\mathscr{G}_1): \quad \begin{array}{ll} \underset{\mathbf{F}_q}{\text{maximize}} & \mathrm{I}_q(\mathbf{F}_q, \mathbf{F}_{-q}) \\ \text{subject to} & \mathbf{F}_q \in \mathscr{F}_q, \end{array} \quad \forall q \in \Omega, \tag{9}$$

where $\Omega \triangleq \{1, \ldots, Q\}$ is the set of players (i.e., the links), $\mathrm{I}_q(\mathbf{F}_q, \mathbf{F}_{-q})$ is the payoff function of player q, given in (7), and $\mathscr{F}_q$ is the set of admissible strategies of player q, defined as

$$\begin{aligned}\mathscr{F}_q \triangleq \big\{\mathbf{F}_q \in\ & \mathbb{C}^{N\times N} : (1/N)\mathsf{Tr}(\mathbf{F}_q\mathbf{F}_q^H) \leq P_q, \\ & \left[\mathbf{W}^H\mathbf{F}_q\mathbf{F}_q^H\mathbf{W}\right]_{kk} \leq \overline{p}_q^{\max}(k), \forall k = 1, \cdots, N\big\}.\end{aligned} \tag{10}$$

The solutions to (9) are the well-known Nash Equilibria [1]. Observe that, for the payoff functions defined in (7), we can limit ourselves to adopt pure strategies w.l.o.g., as we did in (9), since every NE of the game is proved to be achievable using pure strategies [7].

2.2. Competitive maximization of transmission rates

The optimality criterion chosen in the previous section requires the use of an ideal Gaussian codebook with a proper covariance matrix. In practice, Gaussian codes are substituted with simple (suboptimal) finite order signal constellations, such as Quadrature Amplitude Modulation (QAM) or Pulse Amplitude Modulation (PAM), and practical (yet suboptimal) coding schemes. Hence, in this section, we focus on the more practical case where the information bits are mapped onto constellations of finite size, and consider the optimization of the transceivers $(\mathbf{F}_q, \mathbf{G}_q)_{q\in\Omega}$, in order to maximize the transmission rate on each link, under constraints **Co.1** $\div$ **Co.3**.

Given the signal model in (2), where now each vector $\mathbf{s}_q \triangleq (s_q(k))_{k=1}^N$ is drawn from a set of finite-constellations $(\mathcal{C}_{k,q})_{k=1}^N$, i.e., $s_q(k) \in \mathcal{C}_{k,q}$, the transmission rate of each link is trivially defined as the number of transmitted bits per symbol, i.e., $r_q = \sum_{k=1}^N \log_2(|\mathcal{C}_{k,q}|)$, where $|\mathcal{C}_{k,q}|$ denotes the size of the constellation $\mathcal{C}_{k,q}$. In [7], we proved that, under the Gaussian assumption on MUI, the optimal linear receiver $\mathbf{G}_q$ for each user q, given the set of precoding matrices $(\mathbf{F}_q)_{q\in\Omega}$ and the constellations $(\mathcal{C}_{k,q})_{k=1,q\in\Omega}^N$, is the well-known Wiener filter, defined in (8). Using such a $\mathbf{G}_q$, and assuming continuous bit distribution obtained by the gap approximation [11, 12], the optimal set of precoding matrices $\{\mathbf{F}_q\}$ becomes the solution of the following strategic noncooperative game [7]:

$$(\mathscr{G}_2): \quad \begin{array}{ll} \underset{\mathbf{F}_q}{\text{maximize}} & \mathrm{r}_q(\mathbf{F}_q, \mathbf{F}_{-q}) \\ \text{subject to} & \mathbf{F}_q \in \mathscr{F}_q, \end{array} \quad \forall q \in \Omega, \tag{11}$$

where $\mathscr{F}_q$ is given in (10) and the transmission rate $\mathrm{r}_q(\mathbf{F}_q, \mathbf{F}_{-q})$ is defined as

$$\mathrm{r}_q(\mathbf{F}_q, \mathbf{F}_{-q}) = \frac{1}{N}\sum_{k=1}^N \log_2\left(1+\frac{\mathrm{SINR}_{k,q}(\mathbf{F}_q, \mathbf{F}_{-q})}{\Gamma_q}\right), \tag{12}$$

with $\Gamma_q \geq 1$ denoting the gap (which depends only on the constellations and on $P_{e,q}^{\star}$ [11, 12]) and

$$\mathrm{SINR}_{k,q}(\mathbf{F}_q, \mathbf{F}_{-q}) = \frac{1}{\left[(\mathbf{I} + \mathbf{F}_q^H\mathbf{H}_{qq}^H\mathbf{R}_{-q}^{-1}\mathbf{H}_{qq}\mathbf{F}_q)^{-1}\right]_{kk}} - 1. \tag{13}$$

As in (9), in the following we focus on pure strategies only.

3. OPTIMALITY OF DIAGONAL TRANSMISSION

The following Theorem provides the optimal structure of precoding matrices $(\mathbf{F}_q)_{q\in\Omega}$ for both games $\mathscr{G}_1$ and $\mathscr{G}_2$ [7].

Theorem 1 *An optimal solution to the matrix-valued games $\mathscr{G}_1$ and $\mathscr{G}_2$ is*

$$\mathbf{F}_q = \mathbf{W}\sqrt{\mathrm{diag}(\mathbf{p}_q)}, \quad \forall q \in \Omega, \tag{14}$$

where $\mathbf{p} \triangleq (\mathbf{p}_q)_{q\in\Omega}$, with $\mathbf{p}_q \triangleq (p_q(k))_{k=1}^N$, is solution to the vector-valued game $\mathscr{G}$, defined as

$$(\mathscr{G}): \quad \begin{array}{ll} \underset{\mathbf{p}_q}{\text{maximize}} & R_q(\mathbf{p}_q, \mathbf{p}_{-q}) \\ \text{subject to} & \mathbf{p}_q \in \mathscr{P}_q \end{array}, \qquad \forall q \in \Omega, \tag{15}$$

where $R_q(\mathbf{p}_q, \mathbf{p}_{-q})$ and $\mathscr{P}_q$ are the payoff function and the set of admissible strategies of user q, respectively, defined as

$$R_q(\mathbf{p}_q, \mathbf{p}_{-q}) = \frac{1}{N}\sum_{k=1}^N \log\left(1 + \frac{1}{\Gamma_q}\,\mathsf{sinr}_q(k)\right), \tag{16}$$

and

$$\mathscr{P}_q \triangleq \left\{\mathbf{p}_q \in \mathbb{R}^N : \frac{1}{N}\sum_{k=1}^N p_q(k) \le 1,\ 0 \le p_q(k) \le p_q^{\max}(k),\ \forall k\right\}, \tag{17}$$

with $p_q^{\max}(k) \triangleq \overline{p}_q^{\max}(k)/P_q$,

$$\mathsf{sinr}_q(k) \triangleq \frac{|H_{qq}(k)|^2\, p_q(k)}{1 + \sum_{r\neq q} |H_{rq}(k)|^2\, p_r(k)}, \tag{18}$$

where $H_{rq}(k) \triangleq \bar{H}_{rq}(k)\sqrt{P_r/\left(\sigma_q^2\, d_{rq}^{\gamma}\right)}$, and $\Gamma_q = 1$ if $\mathscr{G}_1$ is considered.

According to Theorem 1, a NE of both games $\mathscr{G}_1$ and $\mathscr{G}_2$ is reached using, for each user, a diagonal transmission strategy through the channel eigenmodes (i.e., the frequency bins), irrespective of the channel realizations, power budget, spectral mask constraints and MUI. This result simplifies the original matrix-valued optimization problems (9) and (11), as the number of unknowns for each user reduces from N^2 (the original matrix $\mathbf{F}_q$) to N [the power allocation vector $\mathbf{p}_q = (p_q(k))_{k=1}^N$], with no performance loss. It is straightforward to see that a NE of both matrix-valued games exists if the solution set of $\mathscr{G}$ is non empty. Moreover, the Nash equilibria of $\mathscr{G}$, if they exist, must satisfy the waterfilling solution *for each* user, i.e., the following system of *nonlinear* equations: $\forall q \in \Omega$,

$$\mathbf{p}_q^\star = \mathsf{WF}_q\left(\mathbf{p}_1^\star, \ldots, \mathbf{p}_{q-1}^\star, \mathbf{p}_{q+1}^\star, \ldots, \mathbf{p}_Q^\star\right) = \mathsf{WF}_q(\mathbf{p}_{-q}^\star), \tag{19}$$

with the waterfilling operator $\mathsf{WF}_q(\cdot)$ defined as:

$$[\mathsf{WF}_q(\mathbf{p}_{-q})]_k \triangleq \left[\mu_q - \Gamma_q\,\frac{1 + \sum_{r\neq q}|H_{rq}(k)|^2\, p_r(k)}{|H_{qq}(k)|^2}\right]_0^{p_q^{\max}(k)} \tag{20}$$

with $k \in \{1, \ldots, N\}$. In (20), $[x]_a^b$ denotes the Euclidean projection of x onto the interval $[a, b]$. The water-level μ_q is chosen to satisfy the power constraint $(1/N)\sum_{k=1}^N p_q^\star(k) = 1$.

The full characterization of the game $\mathscr{G}$ in terms of conditions for the existence and uniqueness of a NE [solution to (19)] along with distributed algorithms able to converge to these equilibria are given in the forthcoming sections.

4. EXISTENCE AND UNIQUENESS OF THE NE

Before providing conditions for the uniqueness of the NE of game $\mathscr{G}$, we introduce the following intermediate definitions. Let $\mathcal{D}_q^{\min} \subseteq \{1, \cdots, N\}$ denote the set $\{1, \ldots, N\}$ deprived of the carrier indices that user q would never use as the best response set to any strategies adopted by the other users, for the given set of transmit power and propagation channels [7]:

$$\mathcal{D}_q^{\min} \triangleq \left\{k \in \{1, \ldots, N\} : [\mathsf{WF}_q(\mathbf{p}_{-q})]_k \neq 0 \text{ for some } \mathbf{p}_{-q} \in \mathscr{P}_{-q}\right\}, \tag{21}$$

with $\mathsf{WF}_q(\cdot)$ defined in (20) and $\mathscr{P}_{-q} \triangleq \mathscr{P}_1 \times \cdots \times \mathscr{P}_{q-1} \times \mathscr{P}_{q+1} \times \cdots \times \mathscr{P}_Q$. In [7], we suggested an iterative procedure to obtain a set $\mathcal{D}_q$ such that $\mathcal{D}_q^{\min} \subseteq \mathcal{D}_q \subseteq \{1, \cdots, N\}$. We also introduce the matrix $\mathbf{S}(k) \in \mathbb{R}^{Q\times Q}$, defined as

$$[\mathbf{S}(k)]_{qr} \triangleq \begin{cases} \Gamma_q\,\dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\dfrac{d_{qq}^{\alpha}}{d_{rq}^{\alpha}}\dfrac{P_r}{P_q}, & \text{if } k \in \mathcal{D}_q \cap \mathcal{D}_r \text{ and } r \neq q, \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

where each set $\mathcal{D}_q$ can be chosen as any subset of $\{1, \cdots, N\}$ such that $\mathcal{D}_q^{\min} \subseteq \mathcal{D}_q \subseteq \{1, \cdots, N\}$, with $\mathcal{D}_q^{\min}$ defined in (21). The study of game $\mathscr{G}$ is addressed in the following.

Theorem 2 ([7]) *Game $\mathscr{G}$ admits a nonempty solution set for any given set of channels, spectral mask constraints and transmit power of the users. Furthermore, the NE is unique if*

$$\rho(\mathbf{S}(k)) < 1, \qquad \forall k \in \{1, \ldots, N\}, \tag{C1}$$

where $\mathbf{S}(k)$ is defined in (22) and $\rho(\mathbf{S}(k))$ denotes the spectral radius of $\mathbf{S}(k)$.

We provide now alternative sufficient conditions for Theorem 2. To this end, we first introduce the matrix $\mathbf{S}^{\max} \in \mathbb{R}^{Q\times Q}$, defined as

$$[\mathbf{S}^{\max}]_{qr} \triangleq \begin{cases} \Gamma_q \displaystyle\max_{k\in\mathcal{D}_q\cap\mathcal{D}_r} \dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\dfrac{d_{qq}^{\alpha}}{d_{rq}^{\alpha}}\dfrac{P_r}{P_q}, & \text{if } r \neq q, \\ 0, & \text{otherwise}, \end{cases} \tag{23}$$

with the convention that the maximum in (23) is zero if $\mathcal{D}_q \cap \mathcal{D}_r$ is empty. Then, we have the following corollary of Theorem 2 [7].

Corollary 1 *A sufficient condition for (C1) is:*

$$\rho(\mathbf{S}^{\max}) < 1, \tag{C2}$$

where $\mathbf{S}^{\max}$ is defined in (23).

Remark 1- Physical interpretation of uniqueness conditions: As expected, the uniqueness of NE is ensured if the links are sufficiently far apart from each other. In fact, from (C1), (C2), one can show that there exists a minimum distance beyond which the uniqueness of NE is guaranteed, corresponding to the maximum level of interference that may be tolerated by the users (see also [7] for more intuitive but stronger sufficient conditions). This result agrees with the fact that, as the MUI becomes negligible, the rates R_q in (16) become decoupled and then the rate-maximization problem in (15) for each user admits a unique solution. But, the most interesting result coming from conditions (C1), (C2) is that the uniqueness of the equilibrium is robust against the worst normalized channels $|H_{rq}(k)|^2/|H_{qq}(k)|^2$; in fact, the subchannels corresponding to the highest ratios $|H_{rq}(k)|^2/|H_{qq}(k)|^2$ (and, in particular, the subchannels where $|H_{qq}(k)|^2$ is vanishing) do not necessarily affect the uniqueness, as their carrier indices may not belong to the set $\mathcal{D}_q^{\min}$. This desired property is also confirmed by the numerical example shown in Fig. 1, where we tested the range of validity of uniqueness condition (C1) over a set of channel impulse responses generated as vectors composed of i.i.d. complex Gaussian random variables with zero mean and unit variance. More specifically, in the figure we plot the probability that conditions (C1) is satisfied versus the ratio d_{rq}/d_{qq} (which measures how far apart the links are from each other), for a network composed of $Q = 15$ active links. We tested our conditions considering the set $\mathcal{D}_q$ obtained using the algorithm given in [7]. We can see, from Fig. 1, that the probability that the NE is unique increases as the links become more and more separated of each other (i.e., the ratio d_{rq}/d_{qq} increases). Furthermore, we can verify that

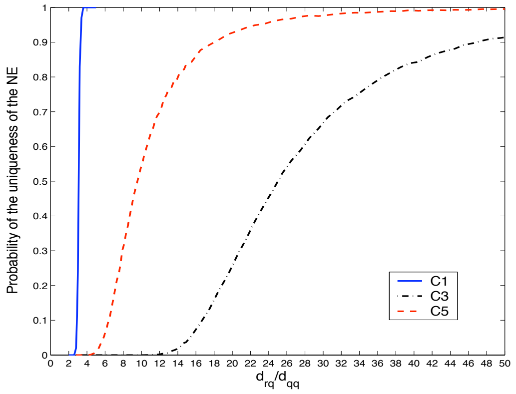

Fig. 1. Probability of (C1), (C3) and (C5) versus d_{rq}/d_{qq}; $Q = 15$, $\gamma = 2.5$, $\Gamma_q = 1$, $d_{rq}/d_{qq} = d_{qr}/d_{rr}$, $\forall r, q \in \Omega$, $N = 64$.

condition (C1) exhibits a neat threshold behavior since it transits very rapidly from the non-uniqueness guarantee to the almost certain uniqueness, as the inter-user distance ratio d_{rq}/d_{qq} increases by a small percentage. This shows that the uniqueness condition (C1) depends, fundamentally, on the interlink distance rather than on the channel realization.

Remark 2 - Comparison with previous conditions: Theorem 2 unifies and generalizes many existence and uniqueness results obtained in the literature [2]-[6] for special cases of game $\mathscr{G}$ in (15). Specifically, in [2]-[4] a game as in (15) is studied, where all the players are assumed to have the same power budget and no spectral mask constraints are considered [i.e., $p_q^{\max}(k) = +\infty, \forall k, q$]. Under this assumptions, the authors proved that the NE of such a game exists and is unique provided that the following condition is satisfied [2, 3]: $\forall\, r, q \neq r \in \Omega$,

$$\Gamma_q \max_{k\in\{1,\ldots,N\}} \left\{ \frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \right\} \frac{d_{qq}^{\alpha}}{d_{rq}^{\alpha}} \frac{P_r}{P_q} < \frac{1}{Q-1}. \qquad \text{(C3)}$$

In the special case of flat-fading channels (i.e., $\bar{H}_{rq}(k) = \bar{H}_{rq}$, $\forall r, q$), the authors in [6] proved that the NE of the game in [2, 3] is unique under the following condition

$$\Gamma_q \sum_{r=1, r\neq q}^{Q} \frac{|\bar{H}_{rq}|^2}{|\bar{H}_{qq}|^2} \frac{d_{qq}^{\alpha}}{d_{rq}^{\alpha}} \frac{P_r}{P_q} < 1, \qquad \forall q \in \Omega. \qquad \text{(C4)}$$

It is straightforward to see that our condition (C1) has a broader validity than both (C3) and (C4) [7, Corollary 3], derived under strong constraints, incorporating, e.g., spectral masks. Recently, alternative sufficient conditions for the uniqueness of the NE of game $\mathscr{G}$ were given in [5]. Among all, an easy condition to be checked is the following:

$$\mathbf{I} + \mathbf{H}(k) \text{ is positive definite for all } k \in \{1, \ldots, N\}, \qquad \text{(C5)}$$

where $\mathbf{H}(k)$ is defined as in (22), with each $\mathcal{D}_q = \{1, \ldots, N\}$.

In Fig. 1, we compare our condition (C1) with (C3) and (C5). We can check that (C1) has a much higher probability of being satisfied than (C3) and (C5). As an example, for a probability 0.99 of guaranteeing the uniqueness of the NE, condition (C1) requires $d_{rq}/d_{qq} \simeq 4$ whereas conditions (C3) and (C5) require $d_{rq}/d_{qq} > 50$ and $d_{rq}/d_{qq} \simeq 50$, respectively. Furthermore, this difference increases as the number Q of links increases [7].

5. ASYNCHRONOUS IWFA

The NE points of the game $\mathscr{G}$ in (15) can be computed using the asynchronous IWFA, proposed in [10]. The algorithm is an instance of the totally asynchronous scheme of [9]: all the users maximize their own rate (16) in a totally asynchronous way via the single user waterfilling solution (20). According to this asynchronous procedure, some users are allowed to update their strategy *more frequently* than the others, and they might perform these updates using *outdated* information on the interference caused from the others. Interestingly, whatever the asynchronous mechanism is, such a procedure converges to a stable NE of the game $\mathscr{G}$, under the following mild conditions on the multiuser interference [10].

Theorem 3 *Assume that condition (C2) of Corollary 1 is satisfied. Then, the asynchronous IWFA of [10] asymptotically converges to the unique NE of game $\mathscr{G}$, for any set of initial feasible conditions and updating schedule.*

Remark 3: Distributed nature and robustness of AIWFA. Since the asynchronous IWFA is based on the waterfilling solution in (20), it can be implemented in a distributed way, since each user, to maximize its own rate, only needs to locally measure the PSD of the interference-plus-noise (see (18)) and waterfill over this level. More interestingly, all the algorithms resulting as special cases of the asynchronous IWFA, such as for example the sequential IWFA [2, 3, 8] or the simultaneous IWFA [8], are guaranteed to reach the unique NE of the game $\mathscr{G}$, under the *same* set of convergence conditions (cf. Theorem 3). By direct product of this generalized framework, one infer that the convergence for these two algorithms is robust to situations where some users may fail to follow the sequential or simultaneous scheduling of updates. What is affected in this case is only the convergence time.

6. CONCLUSIONS

In this paper, we have formulated the problem of finding the optimal structure of linear transceivers in a multipoint-to-multipoint wideband network, as a strategic non-cooperative game. We first considered the theoretical problem of maximizing mutual information on each link, given constraints on the spectral mask and transmit power. Then, we focused on the competitive maximization of the transmission rate on each link, using finite order constellations, under the same constraints as above plus a constraint on the average error probability. We fully characterized both games by providing a unified expression for the optimal structure of the linear transceivers and deriving conditions for the uniqueness of the NE. Then, we showed that the NE of the game can be reached using a totally asynchronous iterative waterfilling based algorithm.

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