Ordered Eigenvalues of a General Class of Hermitian Random Matrices and Performance Analysis of MIMO Systems

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***Abstract*—In this paper we present a general formulation that unifies the probabilistic characterisation of Hermitian random matrices with a specific structure. Based on a unified expression for the joint pdf, we obtain (i) the joint cdf, (ii) the marginal cdf's, and (iii) the marginal pdf's of the ordered eigenvalues, where (ii) and (iii) follow as simple particularizations of (i). Our formulation is shown to include the distribution of some common MIMO channel models such as the uncorrelated and semicorrelated Rayleigh, and the uncorrelated Rician fading MIMO channel, although it is not restricted only to these. Hence, we provide a solid framework for the simultaneous analytical performance analysis of MIMO systems under different channel models. As an example of application, we obtain the exact outage probability of a spatial multiplexing MIMO system transmitting through the strongest channel eigenmodes.**

I. Introduction

Wireless multiple-input multiple-output (MIMO) channels have been recently attracting a great interest since they provide significant improvements in terms of spectral efficiency and reliability with respect to single-input single-output (SISO) channels [1], [2]. Assuming that the communication link has n_{T} transmit and n_{R} receive antennas, the MIMO channel is mathematically described by an $n_{\mathsf{R}} \times n_{\mathsf{T}}$ channel matrix $\mathbf{H}$, whose $(i,j)^{\text{th}}$ entry characterizes the path between the j^{th} transmit and the i^{th} receive antenna. In particular, when communicating over MIMO fading channels, $\mathbf{H}$ is assumed to be drawn from a certain probability distribution, which characterizes the system and scenario of interest and is known as channel model. The system behavior is then evaluated on the average or outage sense, taking into account all possible channel states.

The performance of a MIMO system is usually related to the eigenstructure of $\mathbf{H}$ (channel eigenmodes) or, more exactly, to the non-zero eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ (or $\mathbf{H}^\dagger\mathbf{H}$). Consequently, the probabilistic characterization of these eigenvalues for the adopted channel model is necessary in order to derive analytical expressions for the average and outage performance measures. The common MIMO Rayleigh or Rician fading channels models results in $\mathbf{H}\mathbf{H}^\dagger$ (or $\mathbf{H}^\dagger\mathbf{H}$) being Wishart, Pseudo-Wishart, or Quadratic form distributed.[1] For this reason, the statistical properties of the eigenvalues of these Hermitian random matrices have been widely investigated and effectively applied to analyze the information theoretical limits of MIMO channels [2], [4], [7], [8], [9], [10], [11], [12], [13], [14] as well as the performance of practical MIMO systems [15], [16], [17], [18], [19], [20], [21], [22], [6], [23].

Most of these works deal with the joint pdf of the ordered eigenvalues, the marginal distribution of an unordered eigenvalue or the marginal distribution of the largest and smallest eigenvalue. Nevertheless, an exhaustive analysis of the marginals of all ordered eigenvalues is still missing and this becomes necessary when perfect channel state information (CSI) is available at the transmitter and the weakest channel eigenmodes can be discarded. Some initial contributions in this direction are [20], [22], [23]. In particular, the first order Taylor expansion of the marginal pdf's of all the ordered eigenvalues was given in [20], [23] for the uncorrelated central Wishart distribution and [22] derived the exact and the first order Taylor expansion of the marginal pdf's of the ordered eigenvalues for the uncorrelated noncentral Wishart distribution.

In this paper we present a general formulation that unifies the probabilistic characterisation of Hermitian random matrices with a specific structure. Based on this unified expression for the joint pdf, we obtain (i) the joint cdf, (ii) the marginal cdf's, and (iii) the marginal pdf's of the ordered eigenvalues, where (ii) and (iii) follow as simple particularizations of (i). Then, in order to illustrate the utility of our unified approach, we particularize these results for uncorrelated and correlated central Wishart, correlated central Pseudo-Wishart, and uncorrelated noncentral Wishart matrices. To the best of the authors' knowledge, the joint cdf was unknown for all these distributions and the marginal cdf's and pdf's were only available for the uncorrelated central and noncentral Wishart distributions. Recently, other unified treatments have been proposed in [17], [12], including, however, only uncorrelated and

[1] The more general complex Quadratic form distributions [3], [4], [5], [6] include the Wishart and Pseudo-Wishart distributions as particular cases.

correlated central Wishart and uncorrelated noncentral Wishart matrices. Furthermore, only the distribution of the largest and the smallest eigenvalue and of an unordered eigenvalue were derived.

The rest of the paper is outlined as follows. Section II contains the main contribution of this paper, i.e., the derivations of the joint cdf and both the marginal cdf's and pdf's of the ordered eigenvalues of a general class of Hermitian random matrices. In Section III we establish the matching between this class and some important MIMO channel models. As a straightforward example of application of the proposed characterization, the exact outage performance of a MIMO spatial multiplexing scheme is obtained in Section IV. The paper is finally summarized in Section V.

II. Ordered Eigenvalues of a General Class of Random Matrices

In this section we derive the joint cdf and the marginal cdf's of the ordered eigenvalues of a general class of Hermitian random matrices (formalized next in Assumption 1). This general class is shown in Section III to include $\mathbf{H}\mathbf{H}^\dagger$ (or $\mathbf{H}^\dagger\mathbf{H}$) when $\mathbf{H}$ follows some important MIMO channel models. Hence, the results obtained in this section are extremely useful to analyze the performance of MIMO systems as we illustrate in Section IV.

Assumption 1: We consider the class of Hermitian random matrices, for which the joint pdf of its n non-zero ordered eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, can be expressed as

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} |\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \prod_{t=1}^{n} \varphi(\lambda_t) \tag{1}$$

where $\boldsymbol{\iota}$ is a vector of indices and the summation is for all vectors $\boldsymbol{\iota}$ in the set $\mathcal{I}$, $\mathbf{V}(\boldsymbol{\lambda})$ $(n \times n)$ is a Vandermonde matrix (see Appendix) and matrix $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ $(n \times n)$ satisfies

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) \qquad u, v = 1, \ldots, n. \tag{2}$$

The dimension of $\boldsymbol{\iota}$, the set $\mathcal{I}$, the constant $K^{(\boldsymbol{\iota})}$, and the functions $\zeta_u^{(\boldsymbol{\iota})}(\lambda)$ and $\varphi(\lambda)$ depend on the particular distribution of the random matrix.

A. Joint cdf of the Ordered Eigenvalues

We start by deriving the joint cdf of the ordered eigenvalues of the general class of Hermitian random matrices formalized in Assumption 1).

Theorem 1 ([24, App. B.1]): The joint cdf of the ordered eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, of a random Hermitian matrix satisfying Assumption 1 is given by

$$F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \Pr(\lambda_1 \leq \eta_1, \ldots, \lambda_n \leq \eta_n) \tag{3}$$

$$= \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{\boldsymbol{i}\in\mathcal{S}} \frac{1}{\tau(\boldsymbol{i})} \mathcal{T}\{\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i};\boldsymbol{\eta})\} \tag{4}$$

where $(\eta_1 \geq \cdots \geq \eta_n > 0)$,[2] the summation over $\boldsymbol{i} = (i_1, \ldots, i_n)$ is for all $\boldsymbol{i}$ in the set $\mathcal{S}$ defined as[3]

$$\mathcal{S} = \{\boldsymbol{i} \in \mathbb{N}^n |\max(i_{s-1}, s) \leq i_s \leq n,\ i_s \neq r \text{ if } \eta_r = \eta_{r+1}\} \tag{5}$$

and

$$\tau(\boldsymbol{i}) = \prod_{u=1}^{n} \left((1 - \delta_{i_u, i_{u+1}}) \sum_{v=1}^{u} \delta_{i_u, i_v} \right)! \tag{6}$$

where $\delta_{u,v}$ denotes de Kronecker delta. The operator $\mathcal{T}\{\cdot\}$ is defined in the Appendix and the tensor $\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i};\boldsymbol{\eta})$ $(n \times n \times n)$ is defined as

$$[\mathbf{T}^{(\boldsymbol{\iota})}(\boldsymbol{i};\boldsymbol{\eta})]_{u,v,t} = \int_{\eta_{i_{t+1}}}^{\eta_{i_t}} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda \quad u, v, t = 1, \ldots, n. \tag{7}$$

where $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$ (see Assumption 1).

B. Marginal cdf of the k^{th} Largest Ordered Eigenvalue

We now particularize the joint cdf of the ordered eigenvalues given in Theorem 1 to derive the marginal cdf of the k^{th} largest eigenvalue.

Theorem 2 ([24, App. B.2]): The marginal cdf of the k^{th} largest eigenvalue, λ_k, of a random Hermitian matrix satisfying Assumption 1 is given by

$$F_{\lambda_k}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu}\in\mathcal{P}(i)} |\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)| \tag{8}$$

where $\mathcal{P}(i)$ is the set of all permutations $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ of the integers $(1, \ldots, n)$ such that $(\mu_1 < \ldots < \mu_{i-1})$ and $(\mu_i < \ldots < \mu_n)$, matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\boldsymbol{\mu}, i; \eta)]_{u,v} = \begin{cases} \int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & 1 \leq \mu_v < i \\ \int_{0}^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda & i \leq \mu_v \leq n \end{cases} \tag{9}$$

for $u, v = 1, \ldots, n$, and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$.
In the following, we particularize Theorem 2 to obtain a simplified expression for the marginal cdf of the largest and smallest eigenvalues.

Corollary 1 ([24, App. B.3]): The marginal cdf of the largest eigenvalue, λ_1, of a random Hermitian matrix satisfying Assumption 1 is given by

$$F_{\lambda_1}(\eta) = \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})} |\mathbf{F}^{(\boldsymbol{\iota})}(\eta)| \tag{10}$$

where matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\eta)]_{u,v} = \int_{0}^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda \qquad u, v = 1, \ldots, n \tag{11}$$

and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$.

[2]Note that if some $\eta_k = 0$, then $F_{\boldsymbol{\lambda}}(\boldsymbol{\eta}) = 0$ and if $\eta_{k-1} < \eta_k$ then $F_{\boldsymbol{\lambda}}(\eta_1, \ldots, \eta_{k-1}, \eta_k, \ldots, \eta_n) = F_{\boldsymbol{\lambda}}(\eta_1, \ldots, \eta_{k-1}, \eta_{k-1}, \ldots, \eta_n)$.

[3]Note that $i_n = n$ and by definition $i_0 = 0$, $i_{n+1} = n+1$ and $\eta_{n+1} = 0$.

Corollary 2 ([24, App. B.4]): The marginal cdf of the smallest non-zero eigenvalue, λ_n, of a random Hermitian matrix satisfying Assumption 1 is given by

$$F_{\lambda_n}(\eta) = 1 - \sum_{\boldsymbol{\iota}\in\mathcal{I}} K^{(\boldsymbol{\iota})}|\mathbf{F}^{(\boldsymbol{\iota})}(\eta)| \tag{12}$$

where matrix $\mathbf{F}^{(\boldsymbol{\iota})}(\eta)$ $(n \times n)$ is defined as

$$[\mathbf{F}^{(\boldsymbol{\iota})}(\eta)]_{u,v} = \int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda \qquad u,v = 1,\ldots,n \tag{13}$$

and $\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) = \zeta_u^{(\boldsymbol{\iota})}(\lambda)\varphi(\lambda)\lambda^{v-1}$.

Similarly, the marginal pdf of the k^{th} largest eigenvalue can be easily derived from Theorem 2 just by taking the derivative of (8) (see [24, Cor. 3.3]).

III. Particular Cases of Rayleigh and Rician Fading MIMO Channels

In this section we consider the different distributions of $\mathbf{HH}^\dagger$ or $\mathbf{H}^\dagger\mathbf{H}$ that result when $\mathbf{H}$ follows some important cases of the Rayleigh and Rician MIMO channel models. For these cases, we provide the expressions for the parameters describing the general joint pdf of the eigenvalues in Assumption 1 as well as the expressions needed to particularize the results given in Section II. More exactly, we consider the following cases:

Definition 1 (Uncorrelated Rayleigh MIMO channel): The uncorrelated Rayleigh MIMO fading channel model is defined as

$$\mathbf{H} = \mathbf{H}_w \tag{14}$$

where $\mathbf{H}_w$ is a $n_\mathsf{R} \times n_\mathsf{T}$ random channel matrix with i.i.d. zero-mean unit-variance complex Gaussian entries.

Definition 2 (Min-semicorr. Rayleigh MIMO channel): The semicorrelated Rayleigh fading MIMO channel model with correlation at the side with minimum number of antennas is defined as

$$\mathbf{H} = \begin{cases} \mathbf{\Sigma}^{1/2}\mathbf{H}_w & n_\mathsf{R} \le n_\mathsf{T} \\ \mathbf{H}_w\mathbf{\Sigma}^{1/2} & n_\mathsf{R} > n_\mathsf{T} \end{cases} \tag{15}$$

where $\mathbf{\Sigma}$ is the $n \times n$ positive definite correlation matrix with $n = \min(n_\mathsf{T}, n_\mathsf{R})$.

Definition 3 (Max-semicorr. Rayleigh MIMO channel): The semicorrelated Rayleigh fading MIMO channel model with correlation at the side with maximum number of antennas is defined as

$$\mathbf{H} = \begin{cases} \mathbf{H}_w\mathbf{\Sigma}^{1/2} & n_\mathsf{R} \le n_\mathsf{T} \\ \mathbf{\Sigma}^{1/2}\mathbf{H}_w & n_\mathsf{R} > n_\mathsf{T} \end{cases} \tag{16}$$

where $\mathbf{\Sigma}$ is the $m \times m$ positive definite correlation matrix with $m = \max(n_\mathsf{T}, n_\mathsf{R})$.

Definition 4 (Uncorrelated Ricean MIMO channel): The uncorrelated Ricean fading MIMO channel model is defined as

$$\mathbf{H} = \sqrt{\frac{K_c}{K_c+1}}\overline{\mathbf{H}} + \sqrt{\frac{1}{K_c+1}}\mathbf{H}_w \tag{17}$$

where $K_c \in (0,\infty)$ and $\overline{\mathbf{H}}$ is a $n_\mathsf{R} \times n_\mathsf{T}$ deterministic matrix.

Although we restrict here to simple cases of the Rayleigh and Rician fading MIMO channels (for which closed-form expressions can be obtained), other interesting cases are also included in our unified formulation, for instance, the fully correlated Rayleigh fading MIMO channel considered in [6].

A. Uncorrelated Rayleigh Fading MIMO Channel

Consider an uncorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 1, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ defined as

$$\mathbf{W} = \begin{cases} \mathbf{HH}^\dagger & n_\mathsf{R} \le n_\mathsf{T} \\ \mathbf{H}^\dagger\mathbf{H} & n_\mathsf{R} > n_\mathsf{T} \end{cases} \tag{18}$$

follows a complex uncorrelated central Wishart distribution [25], denoted as $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$ and $m = \max(n_\mathsf{T}, n_\mathsf{R})$. Since the non-zero eigenvalues of $\mathbf{HH}^\dagger$ and $\mathbf{H}^\dagger\mathbf{H}$ coincide, we can derive without loss of generality the statistical properties of the non-zero channel eigenvalues by analyzing the eigenvalues of $\mathbf{W}$.

Joint pdf: The joint pdf of the ordered eigenvalues, $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ is given by [25, eq. (95)][8, eq. (10)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{(m-i)!(n-i)!}|\mathbf{V}(\boldsymbol{\lambda})|^2 \prod_{i=1}^{n} e^{-\lambda_i}\lambda_i^{m-n} \tag{19}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A). Identifying terms, the expression in (19) fits the general pdf given in Assumption 1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped) and define the rest of parameters as in Table II.1.

Marginal Distributions: In order to derive the marginal cdf and pdf of the k^{th} largest eigenvalue using the results presented in Section II-B, we only have to particularize

$$\int_{\eta}^{\infty} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda = \Gamma(d(u+v),\eta) \tag{20}$$

$$\int_{0}^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)d\lambda = \gamma(d(u+v),\eta) \tag{21}$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the upper incomplete [26, eq. (6.5.3)] and lower incomplete [26, eq. (6.5.2)] gamma functions.

The marginal cdf of the largest eigenvalue of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$ was initially derived in [27, Thm. 2] and extended to the marginal cdf of the k^{th} largest eigenvalue in [28, eq. (16)]. Recently, the cdf's of the largest eigenvalue were obtained in [15, eq. (18)][16, Cor. 2][17, eq. (5)]. In addition, the pdf's of the largest and smallest eigenvalue were provided in [15, eq. (22)][16, Cor. 3][17, eq. (23)] and [17, eq. (24)], respectively.

B. Min-semicorrelated Rayleigh Fading MIMO Channel

Consider a min-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 2, then the random Hermitian matrix $\mathbf{W}$ $(n \times n)$ in (18) follows a complex correlated central Wishart distribution [25], denoted as $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$,

	$\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{I}_n)$	$\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$	$\mathbf{W} \sim \mathcal{PW}_n(m, \mathbf{0}_n, \mathbf{\Sigma})$
$K^{(\boldsymbol{\iota})}$	$\prod_{i=1}^{n} \frac{1}{(m-i)!(n-i)!}$	$\prod_{i=1}^{n} \frac{1}{\sigma_i^m (m-i)!} \prod_{i<j}^{n} \frac{\sigma_i \sigma_j}{\sigma_j - \sigma_i}$	$\frac{(-1)^{\sum_{i=1}^{m-n}(\iota_i + i)}}{\prod_{i=1}^{n}(n-i)!} \frac{\prod_{i<j}^{m-n}(\sigma_{\iota_j} - \sigma_{\iota_i})}{\prod_{i<j}^{m}(\sigma_j - \sigma_i)}$
$\varphi(\lambda)$	$e^{-\lambda}\lambda^{m-n}$	λ^{m-n}	1
$\zeta_u^{(\boldsymbol{\iota})}(\lambda)$	λ^{u-1}	$e^{-\lambda/\sigma_u}$	$\sigma_{\iota_{d(u+1)}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{d(u+1)}}}$
$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)$	$e^{-\lambda}\lambda^{d(u+v-1)}$	$e^{-\lambda/\sigma_u}\lambda^{d(v)}$	$\sigma_{\iota_{d(u+1)}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{d(u+1)}}} \lambda^{v-1}$
$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$\Gamma(d(u+v), \eta)$	$\sigma_u^{d(v+1)} \Gamma(d(v+1), \eta/\sigma_u)$	$\sigma_{\iota_{d(u+1)}}^{d(v)} \Gamma(v, \eta/\sigma_{\iota_{d(u+1)}})$
$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$\gamma(d(u+v), \eta)$	$\sigma_u^{d(v+1)} \gamma(d(v+1), \eta/\sigma_u)$	$\sigma_{\iota_{d(u+1)}}^{d(v)} \gamma(v, \eta/\sigma_{\iota_{d(u+1)}})$
$d(v)$	$m-n+v-1$	$m-n+v-1$	$m-n+v-1$

Table II.1

PARAMETERS OF THE UNCORRELATED, MIN-SEMICORRELATED, AND MAX-SEMICORRELATED RAYLEIGH FADING MIMO CHANNELS. (DEFINITIONS 1, 2, AND 3).

where $n = \min(n_\mathsf{T}, n_\mathsf{R})$, $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and $\mathbf{\Sigma}$ is the $n \times n$ positive definite correlation matrix.

Joint pdf: The joint pdf of the ordered eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_{n,m}, \mathbf{\Sigma})$ is given by [25, eq. (95)][8, eq. (17)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{\sigma_i^m (m-i)!} \prod_{i<j}^{n} \frac{\sigma_i \sigma_j}{\sigma_j - \sigma_i} |\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \prod_{i=1}^{n} \lambda_i^{m-n} \tag{22}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix), $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = e^{-\lambda_v/\sigma_u} \qquad u, v = 1, \ldots, n. \tag{23}$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ are the eigenvalues of $\mathbf{\Sigma}$ ordered such that $(\sigma_1 > \cdots > \sigma_n > 0)$. Identifying terms, the expression in (22) coincides with the general pdf given in Assumption 1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped) and define the rest of parameters as in Table II.1.

Marginal distributions: In order to derive the marginal cdf and pdf of the k^{th} largest eigenvalue using the results presented in Section II-B, we only have to particularize

$$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_u^{d(v+1)} \Gamma(d(v+1), \eta/\sigma_u) \tag{24}$$

$$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_u^{d(v+1)} \gamma(d(v+1), \eta/\sigma_u) \tag{25}$$

where $\Gamma(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ are the upper incomplete [26, eq. (6.5.3)] and lower incomplete [26, eq. (6.5.2)] gamma functions.

The marginal cdf's of the largest and smallest eigenvalue of $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{0}_{n,m}, \mathbf{\Sigma})$ were recently derived in [7, Thm. 4 (1)][17, eq. (7)][21, eq. (9)] and in [17, eq. (9)][21, eq. (13)], respectively. The corresponding marginal pdf's were obtained in [17, eq. (26)][21, eq. (17)] and in [21, eq. (18)]. To the best of authors' knowledge, the marginal cdf and pdf of the k^{th} largest eigenvalue were not available in the literature.

C. Max-semicorrelated Rayleigh Fading MIMO Channel

Consider a max-semicorrelated Rayleigh fading MIMO channel $\mathbf{H}$ as given in Definition 3, then the random Hermitian matrix $\mathbf{W}$ $(m \times m)$ defined as

$$\mathbf{W} = \begin{cases} \mathbf{H}^\dagger \mathbf{H} & n_\mathsf{R} \leq n_\mathsf{T} \\ \mathbf{H}\mathbf{H}^\dagger & n_\mathsf{R} > n_\mathsf{T} \end{cases} \tag{26}$$

follows a complex correlated central Pseudo-Wishart distribution [29], denoted as $\mathbf{W} \sim \mathcal{PW}_n(m, \mathbf{0}_m, \mathbf{\Sigma})$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$, $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and $\mathbf{\Sigma}$ is the $m \times m$ positive definite correlation matrix.

Joint pdf: The joint pdf of the ordered non-zero eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$,[4] of $\mathbf{W} \sim \mathcal{PW}_n(m, \mathbf{0}_{n,m}, \mathbf{\Sigma})$ is given by [3, eq. (25)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{(n-i)!} \prod_{i<j}^{m} \frac{1}{(\sigma_j - \sigma_i)} |\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \tag{27}$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix) and $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = \begin{cases} \sigma_u^{v-1} & 1 \leq v \leq m-n \\ \sigma_u^{m-n-1} e^{-\lambda_{v-m+n}/\sigma_u} & m-n < v \leq m \end{cases} \tag{28}$$

for $u, v = 1, \ldots, m$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ are the eigenvalues of $\mathbf{\Sigma}$ ordered such that $(\sigma_1 > \cdots > \sigma_m > 0)$. Performing the Laplace expansion (see e.g. [30, Sec. 33]) over the first $m-n$ columns of $\mathbf{E}(\boldsymbol{\lambda})$, it follows that

$$|\mathbf{E}(\boldsymbol{\lambda})| = \sum_{\boldsymbol{\iota} \in \mathcal{I}} (-1)^{\sum_{i=1}^{m-n}(\iota_i + i)} |\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})||\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})| \tag{29}$$

where the summation over $\boldsymbol{\iota} = (\iota_1, \ldots, \iota_m)$ is for all permutation of integers $(1, \ldots, m)$ such that $(\iota_1 < \cdots < \iota_{m-n})$ and $(\iota_{m-n+1} < \cdots < \iota_m)$ and the matrices $\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})$ $((m-n) \times (m-n))$ and $\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})$ $(n \times n)$ are defined as

$$[\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})]_{u,v} = \sigma_{\iota_u}^{v-1} \qquad u, v = 1, \ldots, m-n \tag{30}$$

$$[\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})]_{u,v} = \zeta_u^{(\boldsymbol{\iota})}(\lambda_v) = \sigma_{\iota_{m-n+u}}^{m-n-1} e^{-\lambda_v/\sigma_{\iota_{m-n+u}}} \tag{31}$$

for $u, v = 1, \ldots, n$. Observing that $\mathbf{V}^{(\boldsymbol{\iota})}(\boldsymbol{\sigma})$ is a Vandermonde matrix (see Appendix A), we can finally rewrite the joint pdf as

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \frac{1}{(n-i)!} \prod_{i<j}^{m} \frac{1}{(\sigma_j - \sigma_i)} \sum_{\boldsymbol{\iota} \in \mathcal{I}} (-1)^{\sum_{i=1}^{m-n}(\iota_i + i)} \prod_{i<j}^{m-n} (\sigma_{\iota_j} - \sigma_{\iota_i}) |\mathbf{E}^{(\boldsymbol{\iota})}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})|. \tag{32}$$

[4] There will be also $m-n$ additional zero eigenvalues.

Identifying terms, the expression in (32) coincides with the general pdf given in Assumption 1 by defining the set $\mathcal{I}$ as

$$\mathcal{I} = \{(\iota_1, \dots, \iota_m) = \pi(1, \dots, m) | (\iota_1 < \cdots < \iota_{m-n}) \text{ and } (\iota_{m-n+1} < \cdots < \iota_n)\} \quad (33)$$

where $\pi(\cdot)$ denotes permutation and the rest of parameters as in Table II.1.

Marginal distributions: In order to derive the marginal cdf and pdf of the k^{th} largest eigenvalue using the results presented in Section II-B, we only have to particularize

$$\int_\eta^\infty \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_{\iota_{d(u+1)}}^{d(v)} \Gamma(v, \eta/\sigma_{\iota_{d(u+1)}}) \quad (34)$$

$$\int_0^\eta \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda = \sigma_{\iota_{d(u+1)}}^{d(v)} \gamma(v, \eta/\sigma_{\iota_{d(u+1)}}) \quad (35)$$

where $\Gamma(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ are the upper incomplete [26, eq. (6.5.3)] and lower incomplete [26, eq. (6.5.2)] gamma functions. The marginal cdf of the largest eigenvalue of $\mathbf{W} \sim \mathcal{PW}_n(m, \mathbf{0}_{n,m}, \mathbf{\Sigma})$ was recently derived in [7, Thm. 4 (2)][21, eq. (21)] and the marginal pdf's of the largest and smallest eigenvalue were calculated in [21, eq. (22)] and in [21, eq. (25)], respectively. To the best of authors' knowledge, the marginal cdf and pdf of the k^{th} largest eigenvalue were not available in the literature.

D. Uncorrelated Rician Fading MIMO Channel

Consider a uncorrelated Rician fading MIMO channel as given in Definition 2, then the random Hermitian matrix $\widetilde{\mathbf{W}} = (K_c + 1)\mathbf{W}$, where $\mathbf{W}$ is given in (18), follows a complex uncorrelated noncentral Wishart distribution [25], denoted as $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$, where $n = \min(n_\mathsf{T}, n_\mathsf{R})$ and $m = \max(n_\mathsf{T}, n_\mathsf{R})$, and the non-centrality parameter $\mathbf{\Omega}$ is defined as

$$\mathbf{\Omega} = \begin{cases} K_c \overline{\mathbf{H}}\overline{\mathbf{H}}^\dagger & n_\mathsf{R} \leq n_\mathsf{T} \\ K_c \overline{\mathbf{H}}^\dagger \overline{\mathbf{H}} & n_\mathsf{R} > n_\mathsf{T} \end{cases}. \quad (36)$$

Note that in this case the non-zero channel eigenvalues, i.e., the eigenvalues of $\mathbf{W}$, are a scaled version of the eigenvalues of $\widetilde{\mathbf{W}}$.

Joint pdf: The joint pdf of the ordered eigenvalues, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, of $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$ is given by [25, eq. (102)] [16, eq. (45)]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \frac{e^{-\sum_{i=1}^n \omega_i}}{((m-n)!)^n} \prod_{i<j}^n \frac{1}{(\omega_j - \omega_i)} |\mathbf{E}(\boldsymbol{\lambda})||\mathbf{V}(\boldsymbol{\lambda})| \prod_{i=1}^n e^{-\lambda_i} \lambda_i^{m-n} \quad (37)$$

where $\mathbf{V}(\cdot)$ is a Vandermonde matrix (see Appendix A) and $\mathbf{E}(\boldsymbol{\lambda})$ is defined as

$$[\mathbf{E}(\boldsymbol{\lambda})]_{u,v} = {}_0F_1(m-n+1; \omega_u \lambda_v) \quad u, v = 1, \dots, n \quad (38)$$

where ${}_0F_1(\cdot; \cdot)$ is a generalized hypergeometric function (see [31, eq. (9.14.1)]) and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ are the eigenvalues of $\mathbf{\Omega}$ ordered such that $(\omega_1 > \cdots > \omega_n > 0)$. Identifying terms, the expression in (37) fits the general pdf given in Assumption 1 if we let $\mathcal{I}$ be a singleton (the superindex $(\boldsymbol{\iota})$ can then be dropped) and define the rest of parameters as in Table III.2.

Marginal distributions: In order to derive the marginal cdf and pdf of the k^{th} largest eigenvalue using the results presented in Section II-B, we only have to particularize

$$\int_0^\eta \xi_{u,v}(\lambda) d\lambda = \int_0^\eta {}_0F_1(m-n+1; \omega_u \lambda) e^{-\lambda} \lambda^{d(v)} d\lambda \quad (39)$$

$$\int_\eta^\infty \xi_{u,v}(\lambda) d\lambda = \int_\eta^\infty {}_0F_1(m-n+1; \omega_u \lambda) e^{-\lambda} \lambda^{d(v)} d\lambda. \quad (40)$$

Using [26, eq. (9.6.47)], it holds that

$$\int_\eta^\infty \xi_{u,v}(\lambda) d\lambda = \frac{e^{\omega_u} 2^{1-v} (m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}} Q_{d(2v), m-n}\left(\sqrt{2\omega_u}, \sqrt{2\eta}\right) \quad (41)$$

where $Q_{m,n}(\cdot, \cdot)$ is the Nuttall Q function [32, eq. (4.104)]. Similarly, using [31, eq. (6.643.2)] and [31, eq. (9.220.2)], it follows that

$$\int_0^\infty \xi_{u,v}(\lambda) d\lambda = \Gamma(d(v+1))_1F_1(d(v+1); m-n+1; \omega_u) \quad (42)$$

and we have that

$$\int_0^\eta \xi_{u,v}(\lambda) d\lambda = \int_0^\infty \xi_{u,v}(\lambda) d\lambda - \int_\eta^\infty \xi_{u,v}(\lambda) d\lambda. \quad (43)$$

The marginal cdf of the k^{th} largest eigenvalue of $\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$ was initially derived in [33, eq. (9)] in terms of an infinite series of determinants. Recently, the marginal cdf the k^{th} largest eigenvalue was obtained in terms of a finite sum of determinants in [22, Thm. 3] and the particular cases of the largest and smallest eigenvalue in [16, Thm. 1][22, Thm. 2] and in [22, Thm. 1], respectively. In addition, the marginal pdf of the maximum eigenvalue was given in [16, Cor. 3] and the case of $\mathbf{\Omega}$ being rank 1 was considered in [16, Cor. 3].

IV. Outage Probability of Spatial Multiplexing MIMO Systems with CSI

As an illustrative application for the joint and marginal cdf's of the ordered eigenvalues given in Section II, we analyze in this section the outage probability of a spatial multiplexing MIMO system with perfect CSI at both sides of the link, which transmits independent substreams through the channel eigenmodes.

A. Signal Model

Following the singular value decomposition (SVD), the channel matrix $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger \quad (44)$$

	$\widetilde{\mathbf{W}} \sim \mathcal{W}_n(m, \mathbf{\Omega}, \mathbf{I}_n)$
$K^{(\boldsymbol{\iota})}$	$\frac{e^{-\sum_{i=1}^{n} \omega_i}}{((m-n)!)^n} \prod_{i<j}^{n} \frac{1}{(\omega_j - \omega_i)}$
$\varphi(\lambda)$	$e^{-\lambda}\lambda^{m-n}$
$\zeta_u^{(\boldsymbol{\iota})}(\lambda)$	${}_0F_1(m-n+1; \omega_u \lambda)$
$\xi_{u,v}^{(\boldsymbol{\iota})}(\lambda)$	${}_0F_1(m-n+1; \omega_u \lambda) e^{-\lambda} \lambda^{d(v)}$
$\int_0^{\eta} \xi_{u,v}^{(\boldsymbol{\iota})}(\lambda) d\lambda$	$d(v)!\,{}_1F_1(d(v+1); m-n+1; \omega_u) - \frac{e^{\omega_u} 2^{1-v} (m-n)!}{\left(\sqrt{2\omega_u}\right)^{m-n}} Q_{d(2v),m-n}\left(\sqrt{2\omega_u}, \sqrt{2\eta}\right)$
$\int_{\eta}^{\infty} \xi^{(\boldsymbol{\iota})} i_{u,v}(\lambda) d\lambda$	$\frac{e^{\omega_u} 2^{1-v} (m-n)!}{\left(\sqrt{2\phi_u}\right)^{m-n}} Q_{d(2v),m-n}\left(\sqrt{2\omega_u}, \sqrt{2\eta}\right)$
$d(v)$	$m-n+v-1$

Table III.2
PARAMETERS OF THE UNCORRELATED RICEAN FADING MIMO CHANNEL (DEFINITION 4).

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and $\sqrt{\mathbf{\Lambda}}$ is a diagonal matrix containing the eigenvalues of $\mathbf{HH}^\dagger$. This way, the channel matrix is effectively decomposed into $\mathrm{rank}(\mathbf{H}) = \min\{n_\mathsf{T}, n_\mathsf{R}\}$ independent orthogonal modes of excitation, which are referred to as channel eigenmodes [1], [2].

Then, assuming perfect CSI at both sides of the link, the channel can be diagonalized and $K \leq \min\{n_\mathsf{T}, n_\mathsf{R}\}$ independent data symbols $\{s_k\}_{k=1,\ldots,K}$ can be communicated simultaneously and without interference as (see [23, Sec. III])

$$\hat{s}_k = \sqrt{\lambda_k \phi_k \mathsf{snr}}\, s_k + n_k \qquad k = 1, \ldots, K \tag{45}$$

where $\hat{s}_k$ is the k^{th} estimated data symbol, λ_k is the k^{th} largest eigenvalue of $\mathbf{HH}^\dagger$, ϕ_k defines the channel non-dependent power allocation policy, and $n_k \sim \mathcal{CN}(0,1)$. The transmit power is constrained such that

$$\sum_{k=1}^{K} \phi_k \mathsf{snr} \leq \mathsf{snr} \tag{46}$$

where snr is the SNR per receive antenna. Each substream experiences then an instantaneous SNR given by

$$\rho_k = \lambda_k \phi_k \mathsf{snr} \qquad k = 1, \ldots, K. \tag{47}$$

B. Outage Probability

The outage probability is defined as the probability that the instantaneous SNR, denoted by ρ, falls below a certain threshold $\bar{\rho}$ [32, eq. (1.4)]:

$$\mathsf{P}_{\mathsf{out}}(\bar{\rho}) \triangleq \mathsf{Pr}(\rho \leq \bar{\rho}). \tag{48}$$

Consider the diagonal spatial multiplexing MIMO system in (45) with instantaneous SNR as in (47). The individual outage probability as defined in (48) of the substream transmitted through the k^{th} strongest channel eigenmode is given by

$$\mathsf{P}_{\mathsf{out}}^{(k)}(\mathsf{snr}) \triangleq \mathsf{Pr}(\rho_k \leq \bar{\rho}) = F_{\lambda_k}\left(\frac{\bar{\rho}}{\phi_k \mathsf{snr}}\right) \tag{49}$$

where $F_{\lambda_k}(\cdot)$ denotes the marginal cdf of the k^{th} channel eigenvalue. Under the MIMO channel models presented in Definitions 1 – 4, $F_{\lambda_k}(\cdot)$ can be easily obtained particularizing Theorem 2 with the corresponding expressions given in Tables II.1 and III.2.

In Figure IV.1 we provide the individual outage probability in (49) of the substream transmitted through the 3^{rd} ($k = 3$) channel eigenmode in a spatial multiplexing MIMO system

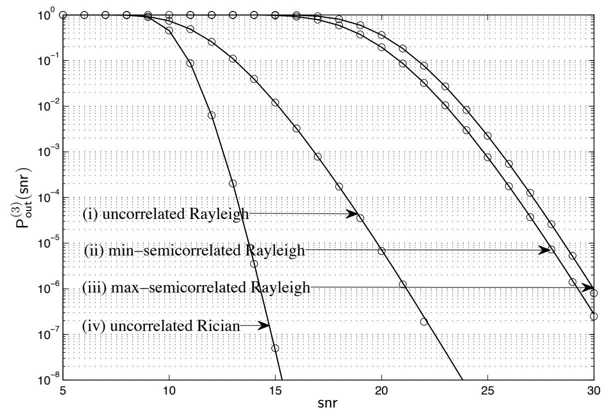

Figure IV.1. Exact (–) and simulated (○) individual outage probability $\mathsf{P}_{\mathsf{out}}^{(k)}(\mathsf{snr})$ for the substream transmitted through the 3rd ($k = 3$) channel eigenmode of an 6×4 MIMO system with $K = 4$, $\phi_k = 1/4$, and using a QPSK modulation.

with $n_\mathsf{T} = 6$ and $n_\mathsf{R} = 4$ antennas when $K = 4$ substreams are established, a QPSK modulation is used on each substream, and the power is uniformly allocated ($\phi_k = 1/4$). The performance threshold has been chosen to guarantee a target BER of $\overline{\mathsf{BER}} = 10^{-3}$ and we have defined the correlation matrix as $[\mathbf{\Sigma}]_{ij} = r^{|i-j|}$ with $r = 0.9$ for the min- and max-semicorrelated Rayleigh, and the rician factor $K_c = 10$ for the uncorrelated Rician fading MIMO channel.

The individual outage probability when transmitting through the strongest eigenmode, i.e., for $K = k = 1$ in (49) and $p_1 = \mathsf{snr}$, has been widely analyzed in the literature, since it corresponds to the outage probability of the beamforming scheme (or maximum ratio transmission [34]). In particular, the outage probability under uncorrelated Rayleigh fading was obtained in [15, Sec. IV][16, Sec. III][18, Sec. II] [19, Sec. IV], under semicorrelated Rayleigh fading in [7, Sec. IV], and under uncorrelated Rician fading in [16, Sec. III]. Additionally, the case of fully correlated Rayleigh fading MIMO channels (not considered explicitly in this paper) has been recently addressed in [6, Sec. IV].

V. CONCLUSIONS

The probabilistic characterization of the eigenvalues of Wishart, Pseudo-Wishart and Quadratic form distributions is critical in the performance evaluation of many communication and signal processing applications. However, the unified perspective provided by this work was missing and can, not only fill the gap of the currently unknown results, but even more importantly, provide a solid framework for the understanding and direct derivation of all the previously derived results and, at the same time, a procedure for the simultaneous analytical performance analysis of MIMO systems under different channel models.

APPENDIX

Definition 5 (Vandermonde matrix [35, eq. (6.1.32)]*):* The $n^{\rm th}$ order Vandermonde matrix in $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, denoted by $\mathbf{V}(\boldsymbol{\lambda})$ $(n \times n)$, is defined as

$$[\mathbf{V}(\boldsymbol{\lambda})]_{i,j} = \lambda_j^{i-1} \qquad i, j = 1, \ldots, n. \tag{50}$$

Lemma 1 (Vandermonde determinant [35, eq. (6.1.33)]*):* The determinant of the $n^{\rm th}$ order Vandermonde matrix introduced in Definition 5 is given by

$$|\mathbf{V}(\boldsymbol{\lambda})| = \prod_{i<j} (\lambda_j - \lambda_i). \tag{51}$$

Definition 6: The operator $\mathcal{T}\{\cdot\}$ over a tensor $\mathbf{T}$ $(n \times n \times n)$ is defined as[5]

$$\mathcal{T}\{\mathbf{T}\} = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \mathrm{sgn}(\boldsymbol{\mu}) \mathrm{sgn}(\boldsymbol{\nu}) \prod_{k=1}^{n} [\mathbf{T}]_{\mu_k, \nu_k, k} \tag{52}$$

where the summation over $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is for all permutations of the integers $(1, \ldots, n)$ and $\mathrm{sgn}(\cdot)$ denotes the sign of the permutation.

[5]Note that this operator was also introduced in [8, Def. 1].

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