Simultaneous Iterative Water-Filling for Gaussian Frequency-Selective Interference Channels

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***Abstract*— The sequential Iterative Water-Filling Algorithm (IWFA) proposed by Yu et al. is by now a popular low-complexity algorithm to compute the Nash equilibrium point of the power allocation game in a Gaussian frequency-selective multiuser interference channel. The algorithm is based on a distributed sequential updating where, at each iteration, the users choose their power allocation, one after the other. However, this sequential updating strategy may slow down its convergence time excessively when the number of users is high. In this paper, we propose an alternative distributed algorithm, called Simultaneous Iterative Water-Filling Algorithm (SIWFA), where at each iteration, all the users update their power allocations simultaneously, rather than sequentially. This reduces the convergence time considerably, specially when the number of users is large. Our main contribution is to provide a unified set of sufficient conditions for the convergence of both IWFA and SIWFA, that are less stringent than those known in the literature for IWFA. These conditions guarantee the convergence of both algorithms also in the presence of spectral mask constraints imposed on the power allocations of the users.**

I. Introduction

The Sequential Iterative Water-Filling Algorithm (IWFA) proposed by Yu et al. in [1] is one of the early dynamic multiuser spectrum optimization techniques aimed at finding the competitive equilibrium (Nash equilibrium) for the rate maximization game in a digital subscriber lines (DSL) system, modeled as a frequency-selective Gaussian interference channel. In this algorithm, each user maximizes its own rate by performing a single-user water-filling, treating the interference from the other users as additive colored noise. At each iteration, the updates occur sequentially, one user after the other. The most appealing features of IWFA are its low-complexity and its distributed nature. In fact, to maximize its own rate, each user needs to measure only the Power Spectral Density (PSD) of the thermal noise plus the received multiuser interference, without requiring specific knowledge of the power allocations adopted by the other users. Furthermore, IWFA was shown to reach interesting performance gains in many practical settings [1], [3].

However, the sequential updating strategy of IWFA becomes a slowing down factor as the number of users increases. Furthermore, IWFA was proved to converge to the (unique) Nash Equilibrium (NE) of the rate maximization game in limited cases, under restrictive constraints on the multiuser interference [1], [2].

To remove these limitations, in this paper we propose a new distributed iterative algorithm, called *Simultaneous Iterative Water-Filling Algorithm* (SIWFA), that falls in the class of the general Jacobi strategies [6]. In SIWFA, all the users update their power allocation *simultaneously*, still according to the single-user water-filling solution, but using the interference generated from the other users in the *previous* iteration. As IWFA, SIWFA is implemented in a distributed way and it achieves the NE of the rate maximization game as well. Furthermore, since in practice there are strict restrictions on the usage of certain frequency bands, specified by the radio spectrum regulatory bodies, we incorporate these constraints in the problem formulation as spectral mask constraints.

Our main contribution is to provide sufficient conditions for the convergence of both IWFA and SIWFA in the presence of spectral mask constraints, that are less stringent than those of [1], [2] (obtained in the absence of spectral constraints), enlarging the range of applications with convergence guarantee.

The paper is organized as follows. Section II gives the system model and formulates the optimization problem as a strategic non-cooperative game. Section III reviews the popular IWFA and Section IV describes the proposed algorithm, SIWFA, along with its convergence properties. In Section V, we provide numerical results showing the superior convergence speeds of SIWFA with respect to IWFA. Finally, in Section VI, the conclusions are drawn.

II. System Model and Problem Formulation

We consider a Gaussian frequency-selective interference channel, composed by Q source terminals and as many destinations. Aiming at finding low-complexity distributed algorithms, we assume no coordination among the users, so that no interference cancelation can be performed at the receiver and multiuser interference is treated as additive colored noise. We also assume that discrete multi-tone modulation and Gaussian codebooks are employed by all the users. This setup arises in many different scenarios, either wireless or wired, as in DSL systems [1] [2], or in wideband wireless meshed networks [3].

Under this system model, we consider the competitive maximization of the information rate *on each link*, given constraints on the average (uncoded) error probability, the transmit power and the spectral emissions of each user. Using the capacity expression for the single user Gaussian channel, the maximum rate achievable on each link is

$$R_q = \frac{1}{N}\sum_{k=0}^{N-1}\log\left(1+\frac{1}{\Gamma_q}\,\mathsf{sinr}_q(k)\right), \qquad (1)$$

where the coefficient $\Gamma_q \geq 1$ is the so-called *SNR-gap*, and $\mathsf{sinr}_q(k)$ denotes the Signal-to-Interference plus Noise Ratio (SINR) on the k-th subcarrier for the q-th link:

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$$\mathsf{sinr}_q(k) = \frac{\mathsf{snr}_q \left|\bar{H}_{qq}(k)\right|^2 p_q(k)}{1+\sum_{r\neq q} \mathsf{inr}_{rq} \left|\bar{H}_{rq}(k)\right|^2 p_r(k)}, \tag{2}$$

where $\bar{H}_{rq}(k) = \sqrt{d_{rq}^{\gamma}}\ H_{rq}(k)$ is the frequency-response of the channel between source r and destination q including the path-loss and the normalized fading process; d_{rq} denotes the distance between source r and destination q, and γ is the path loss exponent; $\mathsf{snr}_q \triangleq P_q/(\sigma_w^2 d_{qq}^{\gamma})$ and $\mathsf{inr}_{rq} \triangleq P_q/(\sigma_w^2 d_{rq}^{\gamma})$ denote the SNR of link q and the Interference-to-Noise Ratio due to the interference received by destination q and generated by source r, with $r \neq q$, respectively; P_q is the transmit power of user q and $p_q(k)$ is the normalized power allocated by the q-th user over the k-th subcarrier, subject to the spectral mask constraints $p_q(k) \leq p_q^{\max}(k), \forall k$,[1] and the power constraint $(1/N)\sum_{k=0}^{N-1} p_q(k) \leq 1$.

We consider a strategic non-cooperative game in which the players are the sources and the payoff functions are their own rates: Each player competes against the others by choosing the PSD that maximizes its own rate, given constraints on the transmit power, the spectral emissions, and the error probability on each link. We use, as optimality criterion, the concept of NE, which is reached when each user, given the power allocation of the other users, does not get any rate increase by moving from the allocation corresponding to the equilibrium. In summary, the structure of the game is the following:

$$\mathscr{G} = \{\Omega, \{\mathscr{P}_q\}_{q\in\Omega}, \{R_q\}_{q\in\Omega}\}, \tag{3}$$

where $\Omega \triangleq \{1, 2, \ldots, Q\}$ denotes the set of the active links, $\mathscr{P}_q$ is the set of admissible power allocation strategies, across the N available sub-carriers, for the q-th player, defined as[2]

$$\mathscr{P}_q = \left\{\mathbf{p}_q \in \mathbb{R}_+^N : 1/N \sum_{k=0}^{N-1} p_q(k) = 1, p_q(k) \leq p_q^{\max}(k), \forall k\right\}, \tag{4}$$

and R_q is given by (1). Observe that the game studied in [1], [2] can be obtained as special case of $\mathcal{G}$ in (3), when no spectral mask constraints are imposed, i.e. $p_q^{\max}(k) = +\infty, \forall k, q$.

The full characterization of the game $\mathscr{G}$ in terms of NEs was given in [3]. Specifically, it was shown in [3] that a (pure strategy) NE always exists for $\mathscr{G}$, and that every NE can be achieved using only pure strategies. Furthermore, sufficient conditions for the uniqueness of the equilibrium were derived in [3], that enlarge all the previous ones available in the literature in the absence of spectral mask constraints (such as [1], [2]).

Once proved that a Nash Equilibrium always exists and under which conditions it is unique, the problem of how to reach such an equilibrium arises. We address this issue in the forthcoming sections.

[1] In order to avoid the trivial solution $p_q(k) = p_q^{\max}(k), \forall k, q$, we assume, for each q, $\sum_k p_q^{\max}(k) > N$.

[2] Observe that, in the feasible strategy set of each player, we can replace, without loss of generality, the original power constraint $(1/N)\ \sum_{k=0}^{N-1} p_q(k) \leq 1$, with $(1/N)\ \sum_{k=0}^{N-1} p_q(k) = 1$, since, at the optimum, this constraint must be satisfied with equality.

III. Sequential Iterative Water-filling Revisited

To compute the NE points of $\mathscr{G}$ in (3) when $p_q^{\max}(k) = +\infty, \forall k, q$, Yu et. al. proposed in [1] a simple iterative water-filling procedure (IWFA) that is an instance of the Gauss-Seidel algorithm [6]. In IWFA, each player, sequentially and according to a fixed order, maximizes its own rate, given the others as interference. It is straightforward to see that, if such a procedure converges, it has to converge to a stable NE of the game $\mathscr{G}$ in (3). In the case of spectral mask constraints, IWFA of [1] can be generalized as follows [3].

Algorithm 1: Sequential Iterative Water-Filling

Set $n = 0$ and $\mathbf{p}_q^{(0)} = \mathsf{WF}_q(\mathbf{p}_{-q})$, with any $\mathbf{p}_{-q} \geq \mathbf{0}$;

`for` $n = 1 : \text{Number\_of\_ iterations}$

  `for` $q = 1 : Q$

$$\mathbf{p}_q^{(n+1)} = \mathsf{WF}_q\left[\mathbf{p}_1^{(n+1)}, \ldots, \mathbf{p}_{q-1}^{(n+1)}, \mathbf{p}_{q+1}^{(n)}, \ldots, \mathbf{p}_Q^{(n)}\right] \tag{5}$$

  `end`

`end`

where $\mathscr{P}_q$ is defined in (4), and $\mathsf{WF}_q[\cdot]$ denotes the water-filling operator, defined as

$$[\mathsf{WF}_q(\mathbf{p}_{-q})]_k \triangleq \left[\mu_q - \mathsf{insr}_q(k)\right]_0^{p_q^{\max}(k)}, \ k = 0, \ldots, N-1, \tag{6}$$

where $\mathbf{p}_{-q} \triangleq [\mathbf{p}_1^T, \ldots, \mathbf{p}_{q-1}^T, \mathbf{p}_{q+1}^T, \ldots, \mathbf{p}_Q]^T$, the symbol $[\cdot]_a^b$, with $b \geq a$ denotes the Euclidean projection on the interval $[a, b]$, and $\mathsf{insr}_q(k)$ is defined as

$$\mathsf{insr}_q(k) \triangleq \Gamma_q \frac{1+\sum_{r\neq q} \mathsf{inr}_{rq} \left|\bar{H}_{rq}(k)\right|^2 p_r(k)}{\mathsf{snr}_q \left|\bar{H}_{qq}(k)\right|^2}. \tag{7}$$

The water-level μ_q in (6) is chosen to satisfy the constraint on the total transmit power. Observe that in the absence of spectral mask constraints (i.e when $p_q^{\max}(k) = +\infty, \forall k, q$), equation (6) becomes the classical water-filling solution, and Algorithm 1 coincides with the IWFA proposed in [1].

Sufficient conditions for the convergence of IWFA to the NE of the game $\mathscr{G}$ in (3) when $p_q^{\max}(k) = +\infty, \forall k, q$, are given in [1] (for $Q = 2$) and [2] (for $Q > 2$) by

$$\max_{k=0,\ldots,N-1} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} < \frac{1}{Q-1}, \quad \forall\ r, q \neq r \in \Omega. \tag{C1}$$

Condition (C1) is also sufficient for the (existence [1]) uniqueness of the equilibrium.

However, in the presence of spectral mask constraints, the results of [1], [2] cannot be used anymore. Our first contribution is to provide sufficient conditions for the convergence of IWFA given in Algorithm 1, in the general case of spectral mask constraints, according to the following.

Theorem 1: IWFA, given in Algorithm 1, converges *geometrically* to the unique NE of the game $\mathcal{G}$ in (3) if one of the following conditions is satisfied

$$\frac{\Gamma_q}{w_q} \sum_{r=1, r\neq q} \max_{k\in\mathcal{D}_r\cap\mathcal{D}_q} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_r < 1, \ \forall q \in \Omega, \tag{C2}$$

$$\frac{1}{w_r}\sum_{q=1,q\neq r}\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\}\frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\frac{P_r}{P_q}\Gamma_q w_q<1,\ \forall r\in\Omega, \tag{C3}$$

where $\boldsymbol{w}\triangleq[w_1,\ldots,w_Q]^T$ is any positive vector, and $\mathcal{D}_q$ is defined as $\mathcal{D}_q\triangleq\{k\in\{0,\ldots,N-1\}:[\mathsf{WF}_q(\mathbf{p}_{-q})]_k\neq 0,$ for some $\mathbf{p}_{-q}\geq\mathbf{0}\}$, with $\mathsf{WF}_q(\cdot)$ given in (6). The set $\mathcal{D}_q$ represents the set $\{0,\ldots,N-1\}$ possibly deprived of the subcarrier indices that user q would never use as the best response set to any strategies used by the other users, for the given set of transmit power and propagation channels. Conditions (C2)-(C3) are also sufficient for the uniqueness of the NE.

Proof: See Appendix. ■

The optimal positive vector $\mathbf{w}$ in (C2)-(C3) can be obtained as a solution of a geometric programming [3].

Remark 1. Note that one can always choose $\mathcal{D}_q=\{0,\ldots,N-1\}$ in (C2)-(C3). However, less stringent conditions are obtained by removing unnecessary sub-carriers, which are never used (cf. Appendix).

Remark 2. Observe that, as expected, the convergence of the algorithm is ensured if the links are sufficiently far apart from each other. Comparing (C2)-(C3) with (C1), we infer that our conditions are larger than (C1); in fact, (C1) implies (C2). But, the most interesting result coming from (C2)-(C3) is that the convergence of IWFA is robust against the worst normalized channels $|H_{rq}(k)|^2/|H_{qq}(k)|^2$; in fact, contrary to what one could infer from (C1), the subchannels corresponding to the highest ratios $|H_{rq}(k)|^2/|H_{qq}(k)|^2$ (and, in particular, the subchannels where $|H_{qq}(k)|^2$ is vanishing) do not necessarily affect the convergence of the algorithm, as their subcarrier indices may not belong to the set $\mathcal{D}_q$.

Remark 3. Observe that IWFA can be implemented in a distributed way, since each user, to maximize its own rate, needs only to measure the PSD of the thermal noise plus the received multiuser interference (see (7)). However, the inner loop in IWFA represents a bottleneck that slows down the whole algorithm when the number of users increases, as we will also show numerically in Section V.

IV. Simultaneous Iterative Water-Filling

The Simultaneous Iterative-Water-Filling Algorithm (SIWFA) proposed in this paper is an instance of the Jacobi scheme [6]: At each iteration, all the users update their own PSD *simultaneously*, performing the single user water-filling solution (6), given the interference generated by the other users in the *previous* iteration. If such a procedure converges, then, by definition, it must converge to a NE of the game. Stated in mathematical terms, we have the following algorithm.

Algorithm 2: Simultaneous Iterative Water-Filling

Set $n=0$ and $\mathbf{p}_q^{(0)}=\mathsf{WF}_q(\mathbf{p}_{-q})$, with any $\mathbf{p}_{-q}\geq\mathbf{0}$;

`for` $n=1:\text{Number\_of\_ iterations}$

$$\mathbf{p}_q^{(n+1)}=\alpha\mathbf{p}_q^{(n)}+(1-\alpha)\mathsf{WF}_q\left[\mathbf{p}_{-q}^{(n)}\right],\quad\forall q\in\Omega; \tag{8}$$

`end`

where $\mathscr{P}_q$ and $\mathsf{WF}_q[\cdot]$ are given in (4) and (6) respectively.

The factor $\alpha\in[0,1)$ can be interpreted as a forgetting factor: The higher is α, the longer is the memory of the algorithm; conversely, when $\alpha=0$, SIWFA has no memory at all and the algorithm becomes a *simultaneous* water-filling. The choice of α depends on the channel stationarity and on possible channel fluctuations or estimation errors [3].

Remark 4. Interestingly, SIWFA is guaranteed to converge to the unique NE of the game, under weaker (sufficient) conditions than those required by IWFA, as given in the following [3].

Theorem 2: SIWFA, given in Algorithm 2, converges *geometrically* to the unique NE of the game $\mathcal{G}$ in (3) if

$$(C4):\qquad\rho\left(\mathbf{H}^T(k)\mathbf{H}(k)\right)<1,\qquad\forall k=0,\ldots,N-1, \tag{9}$$

where $\rho\left(\mathbf{H}^T(k)\mathbf{H}(k)\right)$ denotes the spectral radius of the matrix $\mathbf{H}^T(k)\mathbf{H}(k)$ [5], and $\mathbf{H}(k)$ is defined as

$$[\mathbf{H}(k)]_{qr}\triangleq\begin{cases}\Gamma_q\dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\dfrac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\dfrac{P_r}{P_q}, & \text{if}\ \ k\in\mathcal{D}_q\cap\mathcal{D}_r, q\neq r\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

To give additional insight into the physical interpretation of sufficient conditions for the convergence of SIWFA, we provide the following.

Theorem 3: SIWFA, given in Algorithm 2, converges *geometrically* to the unique NE of the game $\mathcal{G}$ in (3) if conditions (C2)-(C3) of Theorem 1 are satisfied.

Proof: See Appendix. ■

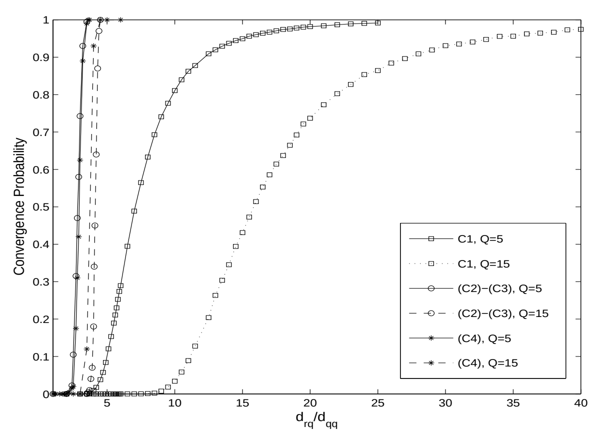

Fig. 1. Probability of (C1), (C2)-(C3) and (C4) versus d_{rq}/d_{qq}; $\gamma=2.5$, $d_{rq}=d_{qr}$, $\Gamma_q=1$, $\forall r,q$.

Since conditions (C1), (C2)-(C3) and (C4) depend on the channel realizations $\{\bar{H}_{rq}(k)\}$ and on the distances $\{d_{rq}\}$, there is a nonzero probability that they are not satisfied for a given setup. In order to compare the goodness of the above conditions, we tested them over a set of channel impulse responses generated as vectors composed of i.i.d. complex Gaussian random variables with zero mean and unit variance. We plot in Figure 1 the probability that conditions (C1), (C2)-(C3) and (C4) are satisfied versus the ratio d_{rq}/d_{qq}. For the sake of simplicity, we assumed $d_{rq}=d_{qr}$, $P_q=P_r$, $\forall r,q$, $\Gamma_q=1$. and $\mathbf{w}$. We considered $Q=5$ (solid lines) and $Q=15$ (dashed lines) active links. We tested our conditions considering in (C2)-(C3) and (C4) the set $\mathcal{D}_q$, as obtained in [3]. We can see, from Figure 1, that the probability that the algorithms converge increases as the links become more and more separated of each other (i.e., the ratio d_{rq}/d_{qq} increases). It is important to remark that the main difference between our conditions and those given in [1], [2] is that the probability that (C2)-(C3) (or (C4)) are satisfied, differently from (C1), exhibits a neat threshold behavior as it passes very rapidly from the non-convergence guarantee to the almost certain convergence situation, as the inter-user distance ratio d_{rq}/d_{qq} increases by a small percentage. Interestingly this threshold does not depend on the particular channel realization, but only on the number of users, the number of subcarriers and the power budget.

As an example, for a system with $Q = 15$ links and probability of guaranteeing convergence of 0.99, conditions (C2)-(C3) only require $d_{rq}/d_{qq} \simeq 4.2$ whereas conditions (C1) require $d_{rq}/d_{qq} > 40$. Furthermore, this difference increases as the number Q of links increases.

Remark 5. Since SIWFA is still based on the water-filling solution (6), it keeps the most appealing features of the IWFA, namely its low-complexity distributed nature. But, thanks to the Jacobi-based update, all the users are allowed to choose their optimal power allocation *simultaneously*, which makes SIWFA faster than IWFA, especially if the number of active users in the network is large (cf. Section V).

V. Numerical Results

In this section, we compare SIWFA and IWFA in terms of convergence speed. In Figure 2, we plot the user rates provided by IWFA and SIWFA, as a function of time. In IWFA, time coincides with the inner iterations, whereas in SIWFA time coincides with the iteration index n. To make the figure not excessively overcrowded, we report only 4 out of 15 curves. As expected, IWFA is shown to be slower than SIWFA, especially if the number Q of active links is large. In fact, as clearly shown in the picture, in the iterative procedure of IWFA, each user, to update its power allocation, is forced to wait all the other users that are scheduled before it. For example, in the system plotted in Figure 2, the user that is declared to be the last one in the iterative procedure, must wait all the other users' updates, i.e. 14 iterations. Instead, with SIWFA all the users update their powers at the same time.

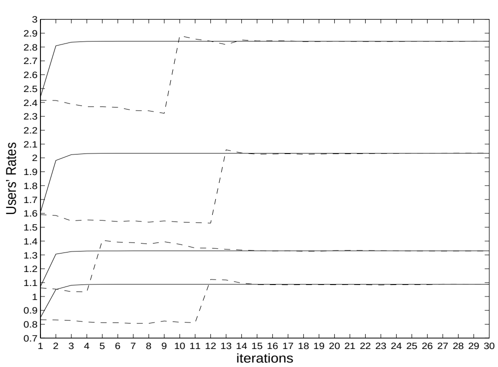

Fig. 2. Users' rates versus iterations: IWFA (dashed-line curves) and SIWFA (solid line curves); $Q = 15$, $\Gamma_q = 1$, $d_{qq} = d_{rr}$, $\gamma = 2.5$, $d_{qr} = d_{rq}$, $P_q = P_r, \forall r, q$, $\mathsf{snr}_q = 10$ dB, and $\mathsf{inr}_{rq} = 5$ dB.

VI. Conclusion

In this paper, we have proposed a new iterative distributed algorithm, called SIWFA, that solves the rate maximization game in frequency-selective multi-user Gaussian interference channel, in the presence of spectral mask constraints. SIWFA is proved to reach the unique NE of the game faster and under weaker (sufficient) conditions than the popular IWFA. Thus SIWFA is an attractive alternative to IWFA. In [7], we generalize the proposed algorithms to the case in which the update of the strategies from the users is performed in a totally asynchronous way.

Appendix: Proof of Theorem 1 and Theorem 3

A complete proof of Theorem 1 and Theorem 3 is given in [3]. Because of the space limitation, here we provide only the stretch of the proof.

To derive sufficient conditions for the convergence of both IWFA and SIWFA, we will use the following.

Lemma 1: The water-filling operator $\mathsf{WF}_q[\cdot]$ in (6) can be equivalently rewritten as

$$\mathsf{WF}_q[\mathbf{p}_{-q}] = [-\mathbf{insr}_q]_{\mathscr{P}_q} \tag{11}$$

where $\mathbf{insr}_q \triangleq [\mathsf{insr}_q(0), \ldots, \mathsf{insr}_q(N-1)]^T$, and $\mathscr{P}_q$ and $\mathsf{insr}_q(k)$ are defined in (4) and (7), respectively, and $[\cdot]_{\mathscr{P}_q}$ denotes the Euclidean projection on $\mathscr{P}_q$.

Proof: The Euclidean projection of the real negative vector $-\mathbf{insr_q}$ on the convex set $\mathscr{P}_q$ is described by the following convex optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \|\mathbf{x} - (-\mathbf{insr_q})\|_2^2 \\ \text{subject to} & 0 \leq x_k \leq p_q^{\max}(k), \quad k = 0, 1, \ldots, N-1, \\ & (1/N) \ \sum_{k=0}^{N-1} x_k = 1. \end{array} \tag{12}$$

Using KKT optimality conditions of (12), we obtain the desired equivalence between (6) and (11) (see [3] for more details). The water-filling based optimal solution of (12) in the case of $p_q^{\max}(k) = +\infty, \forall k, q$, was already given in [4, Lemma 1]. ■

Using Lemma 1, it is convenient to rewrite the water-filling operator in (11) as

$$\mathsf{WF}_q[\mathbf{p}_{-q}] = \left[-\boldsymbol{\sigma}_q - \sum_{r \neq q} \mathbf{H}_{rq}\mathbf{p}_r\right]_{\mathscr{P}_q},$$

where $\mathbf{H}_{rq} \triangleq \Gamma_q(\mathsf{inr}_{rq}/\mathsf{snr}_q)\,\mathrm{diag}\left(\left\{\left|\bar{H}_{rq}(k)\right|^2/\left|\bar{H}_{qq}(k)\right|^2\right\}_k\right)$, and $\boldsymbol{\sigma}_q \triangleq \Gamma_q(1/\mathsf{snr}_q)\left[1/\left|\bar{H}_{qq}(0)\right|^2, \ldots, 1/\left|\bar{H}_{qq}(N-1)\right|^2\right]^T$, $\forall r \neq q$, $q \in \Omega$. In order to provide a unified proof of both Theorem 1 and Theorem 3, we introduce the following mapping. Let be $\mathscr{P}^{\text{wf}} = \mathscr{P}_1^{\text{wf}} \times \cdots \times \mathscr{P}_Q^{\text{wf}} \subseteq \mathscr{P}$, where

$$\mathscr{P}_q^{\text{wf}} \triangleq \left\{\mathbf{p}_q \in \mathscr{P}_q : \mathbf{p}_q = \mathsf{WF}_q[\mathbf{p}_{-q}], \text{ for some } \mathbf{p}_{-q} \geq \mathbf{0}\right\}, \tag{13}$$

is the subset of $\mathscr{P}_q$ containing all the admissible strategies of user q, achievable as water-filling solution over some positive interference profile. For any fixed $\alpha \in [0, 1)$, let $\mathbf{T}(\mathbf{p}) = [\mathbf{T}_1^T(\mathbf{p}), \ldots, \mathbf{T}_Q^T(\mathbf{p})]^T : \mathscr{P}^{\text{wf}} \mapsto \mathscr{P}^{\text{wf}}$ be the mapping defined, for each q and $\mathbf{p} \in \mathscr{P}^{\text{wf}}$, as

$$\mathbf{T}_q(\mathbf{p}) \triangleq \alpha\ \mathbf{p}_q + (1-\alpha)\left[-\boldsymbol{\sigma}_q - \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{p}_r\right]_{\mathscr{P}_q}. \qquad (14)$$

Observe that all the NEs of the game $\mathscr{G}$ in (3) are fixed points of $\mathbf{T}(\mathbf{p})$, whose existence is thus guaranteed by [3, Theorem 2]. Furthermore, IWFA given in Algorithm 1 is an instance of the Gauss-Seidel scheme [6] based on the mapping $\mathbf{T}(\mathbf{p})$ in (14) with $\alpha = 0$, whereas SIWFA in Algorithm 2 is an instance of the Jacobi scheme, still based on $\mathbf{T}(\mathbf{p})$. Hence, according to [6, Prop. 1.4] and [6, Prop. 1.1], the *geometric* convergence of both IWFA and SIWFA to the *unique* NE of $\mathscr{G}$ (i.e. the unique fixed-point of $\mathbf{T}(\mathbf{p})$) is guaranteed if $\mathbf{T}(\mathbf{p})$ is a contraction with respect to the following block-maximum norm $\|\cdot\|$

$$\|\mathbf{T}(\mathbf{p})\| \triangleq \max_{q\in\Omega}\frac{\|\mathbf{T}_q(\mathbf{p})\|_2}{w_q}, \qquad (15)$$

where $\mathbf{w} \triangleq [w_1,\ldots,w_Q]^T$ is any given positive vector.

We prove now that conditions (C2) are sufficient for $\mathbf{T}(\mathbf{p})$ to be a contraction. Given $\mathbf{p}^{(1)} = [\mathbf{p}_1^{(1)T},\ldots,\mathbf{p}_Q^{(1)T}]^T$, $\mathbf{p}^{(2)} = [\mathbf{p}_1^{(2)T},\ldots,\mathbf{p}_Q^{(2)T}]^T \in \mathscr{P}^{\text{wf}}$, define, for each q,

$$e_{Tq} \triangleq \left\|\mathbf{T}_q(\mathbf{p}^{(1)}) - \mathbf{T}_q(\mathbf{p}^{(2)})\right\|_2 \text{ and } e_q \triangleq \left\|\mathbf{p}_q^{(1)} - \mathbf{p}_q^{(2)}\right\|_2. \qquad (16)$$

For any $\mathbf{p}^{(1)}$, $\mathbf{p}^{(2)} \in \mathscr{P}^{\text{wf}}$ and $q \in \Omega$, we have

$$\begin{aligned} e_{Tq} = &\ \left\|\mathbf{T}_q(\mathbf{p}^{(1)}) - \mathbf{T}_q(\mathbf{p}^{(2)})\right\|_2 \\ \overset{(a)}{\leq} &\ \alpha\, e_q + (1-\alpha)\left\|\left[-\boldsymbol{\sigma}_q - \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{p}_r^{(1)}\right]_{\mathscr{P}_q} - \left[-\boldsymbol{\sigma}_q - \sum_{r\neq q}\mathbf{H}_{rq}\mathbf{p}_r^{(2)}\right]_{\mathscr{P}_q}\right\|_2 \\ \overset{(b)}{\leq} &\ \alpha\, e_q + (1-\alpha)\left\|\left(\sum_{r\neq q}\mathbf{H}_{rq}\left(\mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)}\right)_{\mathcal{D}_r}\right)_{\mathcal{D}_q}\right\|_2 \\ \overset{(c)}{=} &\ \alpha\, e_q + (1-\alpha)\left\|\sum_{r\neq q}\overline{\mathbf{H}}_{rq}\left(\mathbf{p}_r^{(1)} - \mathbf{p}_r^{(2)}\right)\right\|_2 \\ \overset{(d)}{\leq} &\ \alpha\, e_q + (1-\alpha)\sum_{r\neq q}\left(\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}[\mathbf{H}_{rq}]_{kk}\right)e_r, \end{aligned} \qquad (17)$$

where: (a) and (d) follow from the triangle inequality [5] (on the fact that $\alpha \geq 0$); in (b) $(\mathbf{x})_{\mathcal{D}_q}$ denotes a vector whose k-th component is equal to x_k if $k \in \mathcal{D}_q$, and 0 otherwise. The inequality follows from the definition of $\mathcal{D}_q$ and the nonexpansive property of the projection operator [6, Prop. 3.2(c)], i.e. $\|[\mathbf{x}]_{\mathscr{P}_q} - [\mathbf{y}]_{\mathscr{P}_q}\|_2 \leq \|\mathbf{x}-\mathbf{y}\|_2$; and (c) follows from the definition of $\mathcal{D}_q$, where $\overline{\mathbf{H}}_{rq}$ is a diagonal matrix defined as

$$[\overline{\mathbf{H}}_{rq}]_{kk} \triangleq \begin{cases} [\mathbf{H}_{rq}]_{kk}, & \text{if } k\in\mathcal{D}_r\cap\mathcal{D}_q, \\ 0, & \text{otherwise.}\end{cases} \qquad (18)$$

Introducing the $Q\times Q$ matrix $\mathbf{H}^{\max}$, defined as

$$[\mathbf{H}^{\max}]_{qr} \triangleq \begin{cases} (1-\alpha)\max\limits_{k\in\mathcal{D}_r\cap\mathcal{D}_q}[\mathbf{H}_{rq}]_{kk}, & r\neq q, \\ \alpha, & r = q,\end{cases} \qquad (19)$$

and the vectors

$$\mathbf{e_T} \triangleq [e_{T_1},\ldots,e_{T_Q}]^T, \quad \text{and} \quad \mathbf{e} \triangleq [e_1,\ldots,e_Q]^T, \qquad (20)$$

with e_{T_q} and e_q defined in (16), the set of inequalities in (17) for all q, can be rewritten in vectorial form as

$$\mathbf{0} \leq \mathbf{e_T} \leq \mathbf{H}^{\max}\mathbf{e}, \quad \forall \mathbf{p}^{(1)},\mathbf{p}^{(2)} \in \mathscr{P}^{\text{wf}}. \qquad (21)$$

Using the weighted maximum vector norm $\|\cdot\|_\infty^{\mathbf{w}}$

$$\|\mathbf{x}\|_\infty^{\mathbf{w}} \triangleq \max_q \frac{|x_q|}{w_q}, \quad \mathbf{w} > \mathbf{0}, \qquad (22)$$

in combination with (21), we have, $\forall \mathbf{p}^{(1)},\mathbf{p}^{(2)} \in \mathscr{P}^{\text{wf}}$ and $\forall \mathbf{w} > \mathbf{0}$:

$$\|\mathbf{e_T}\|_\infty^{\mathbf{w}} \leq \|\mathbf{H}^{\max}\mathbf{e}\|_\infty^{\mathbf{w}} \leq \|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}}\,\|\mathbf{e}\|_\infty^{\mathbf{w}}, \qquad (23)$$

where $\|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}}$ is the weighted maximum matrix norm induced by the vector norm $\|\cdot\|_\infty^{\mathbf{w}}$ in (22) and defined as [5]

$$\|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}} \triangleq \max_q \frac{1}{w_q}\sum_{r=1}^{Q} w_r[\mathbf{H}^{\max}]_{qr}, \quad \mathbf{w} > \mathbf{0}. \qquad (24)$$

Finally, using (15) and (23), we obtain the desired result for the mapping $\mathbf{T}$, as shown next:

$$\begin{aligned} \left\|\mathbf{T}(\mathbf{p}^{(1)}) - \mathbf{T}(\mathbf{p}^{(2)})\right\| &= \|\mathbf{e_T}\|_\infty^{\mathbf{w}} \leq \|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}}\,\|\mathbf{e}\|_\infty^{\mathbf{w}} \\ &= \|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}}\left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|, \end{aligned} \qquad (25)$$

$\forall \mathbf{p}^{(1)},\mathbf{p}^{(2)} \in \mathscr{P}^{\text{wf}}$, $\forall \mathbf{w} > \mathbf{0}$. According to (25), the mapping $\mathbf{T}$ is a block-contraction with respect to the norm $\|\cdot\|$ in (15), if

$$\|\mathbf{H}^{\max}\|_\infty^{\mathbf{w}} < 1, \text{ for some } \mathbf{w} > \mathbf{0}; \qquad (26)$$

which, using (24) and (19), provides (C2) (it is key in this step that $\alpha < 1$).

Convergence of IWFA and SIWFA under conditions (C3), can be obtained from (26), using [6, Corollary 6.1], as proved in [3].

References

[1] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines", *IEEE Jour. on Selected Areas in Communications*, vol. 20, pp. 1105-1115, June 2002.

[2] S. T. Chung, S. J. Kim, J. Lee, and J. M. Cioffi, "A game-theoretic approach to power allocation in frequency-selective Gaussian interference channels", in *Proc. 2003 IEEE Int. Symp. on Information Theory (ISIT 2003)*, p. 316, June 2003.

[3] G. Scutari, D. P. Palomar, and S. Barbarossa, "Optimal Multiplexing Strategies for Wideband Meshed Networks based on Game Theory: Nash Equilibria and Distributed Algorithms", submitted to *IEEE Trans. on Signal Processing*.

[4] D. P. Palomar, "Convex Primal Decomposition for Multicarrier Linear MIMO Transceivers," *IEEE Trans. on Signal Processing*, Vol. 53, No. 12, pp. 4661-4674, Dec. 2005.

[5] R. Horn, and C. R. Johnson, *Matrix Analysis*, Cambridge Univ., 1985.

[6] D. P Bertsekas, *Nonlinear Programming*, Athena Scientific, 2nd Ed.,1999.

[7] G. Scutari, D. P. Palomar, and S. Barbarossa, "Asynchronous Iterative Water-Filling for Gaussain Frequency-Selective Interference Channles: A Unified Framework ", *Proc. 2006 IEEE Workshop on Signal Processing Advances in Wireless Communications, (SPAWC-2006)*, July 2006.