ASYNCHRONOUS ITERATIVE WATER-FILLING FOR GAUSSIAN FREQUENCY-SELECTIVE INTERFERENCE CHANNELS: A UNIFIED FRAMEWORK

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ABSTRACT

In this paper we propose a unified framework, based on a new distributed algorithm to compute the Nash equilibrium point of the power allocation game in a frequency-selective multiuser interference channel. The proposed scheme is based on a *totally asynchronous* updating of the power allocation from the users, where some users may change their power allocation more frequently than others and, furthermore, they are allowed to use also outdated version of the interference. The proposed algorithm contains as special cases the well-known iterative water-filling algorithm, either sequential or simultaneous. Our main contribution is then to provide a unified set of sufficient conditions under which all these algorithms are guaranteed to convergence to the unique Nash equilibrium of the game. These conditions enlarge those existing in the literature for the convergence of the sequential iterative water-filling algorithm.

1. INTRODUCTION

The Sequential Iterative Water-Filling Algorithm (IWFA) proposed by Yu et al. in [1] is one of the early dynamic multiuser spectrum optimization techniques aimed at finding the competitive equilibrium (Nash equilibrium) for the rate maximization game in a digital subscriber lines (DSL) system, modelled as a frequency-selective Gaussian Interference Channel. In this algorithm, each user maximizes its own rate by performing a single-user water-filling, treating the interference generated from the other users as additive colored noise. At each iteration, the updates occur sequentially, one user after the other. The most appealing features of IWFA are its low-complexity and its distributed nature. Furthermore, IWFA was shown to reach interesting performance gains in many practical settings [1, 2].

However, the sequential updating strategy of IWFA becomes a slowing down factor as the number of users increases. Furthermore, IWFA was proved to converge to the (unique) Nash Equilibrium (NE) of the rate maximization game only under restrictive conditions on the multiuser interference [1, 3]. To remove these limitations, in [4] we proposed a new distributed iterative algorithm, called Simultaneous Iterative Water-Filling Algorithm (SIWFA), that falls in the class of the general Jacobi (overrelaxation) strategies [5]. SIWFA was shown to converge to the unique NE of the rate maximization game faster than IWFA and under weaker conditions on the multiuser interference.

In this paper we generalize results of [1, 3, 4] proposing a unified framework, based on a new fully distributed algorithm, called *Asynchronous Iterative Water-Filling Algorithm* (AIWFA), that falls in the class of totally asynchronous schemes of [5]. In AIWFA, all users still update their power allocation water-pouring the interference level generated from the others. But, differently from IWFA and SIWFA, in AIWFA the updates can be performed in a *totally asynchronous* way [5]. This means that some users may choose their Power Spectral Density (PSD) more frequently than others, and, more interestingly, they may use an *outdated* version of the interference caused from the others, without affecting the convergence of the algorithm. These features make AIWFA appealing for all practical scenarios, either wired or wireless, where strong constraints on synchronization cannot be met.

We then derive sufficient conditions for the convergence of AIWFA to the unique NE of the rate maximization game, less stringent than those of [1, 3]. These conditions guarantee also the convergence of both IWFA [1, 3] and SIWFA [4], since they are just two special cases of the more general AIWFA. Interestingly, all the schemes derived by AIWFA, irrespective of their synchronous or asynchronous nature, are guaranteed to converge under the same conditions.

The paper is organized as follows. Section 2 gives the system model and formulates the optimization problem as a strategic noncooperative game. Section 3 describes the proposed algorithm, AIWFA, along with its convergence properties. In Section 4, we review IWFA and SIWFA as special cases of AIWFA and provide new convergence conditions. Finally, Section 5 draws the conclusions.

2. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a Gaussian frequency-selective interference channel, composed by Q source terminals and as many destinations. Aiming at finding low-complexity distributed algorithms, we assume no coordination among the users, so that no interference cancelation can be performed at the receiver and multiuser interference is treated as additive colored noise. We also assume that discrete multi-tone modulation and Gaussian codebooks are employed by all the users. This setup arises in many different scenarios, either wireless or wired, as in DSL systems [1, 3], or in wideband wireless meshed networks [2].

Under this system model, we consider the competitive maximization of the information rate *on each link*, given constraints on the average (uncoded) error probability and on the transmit power for each user. Using the capacity expression for the single user Gaussian channel, the maximum rate achievable on each link is

$$R_q = \frac{1}{N}\sum_{k=0}^{N-1}\log\left(1+\frac{1}{\Gamma_q}\,\mathsf{sinr}_q(k)\right), \tag{1}$$

where the coefficient $\Gamma_q \geq 1$ is the so-called *SNR-gap*, and $\mathsf{sinr}_q(k)$ denotes the Signal-to-Interference plus Noise Ratio (SINR) on the k-th subcarrier for the q-th link:

$$\mathsf{sinr}_q(k) = \frac{\mathsf{snr}_q\left|\bar{H}_{qq}(k)\right|^2 p_q(k)}{1+\sum_{r\neq q}\mathsf{inr}_{rq}\left|\bar{H}_{rq}(k)\right|^2 p_r(k)}, \tag{2}$$

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where $\bar{H}_{rq}(k) = \sqrt{d_{rq}^{\gamma}}\ H_{rq}(k)$ is the frequency-response of the channel between source r and destination q including the path-loss and the normalized fading process; d_{rq} denotes the distance between source r and destination q, with γ the path loss exponent; $\mathsf{snr}_q \triangleq P_q/(\sigma_w^2 d_{qq}^{\gamma})$ and $\mathsf{inr}_{rq} \triangleq P_q/(\sigma_w^2 d_{rq}^{\gamma})$ denote the SNR of link q and the Interference-to-Noise Ratio due to the interference received by destination q and generated by source r, with $r \neq q$, respectively; P_q is the transmit power of user q and $p_q(k)$ is the normalized power allocated by the q-th user over the k-th subcarrier, subject to the power constraint $(1/N)\ \sum_{k=0}^{N-1} p_q(k) \leq 1$.

We consider a strategic non-cooperative game in which the players are the sources and the payoff functions are their own rates: Each player competes against the others by choosing the PSD that maximizes its own rate, given constraints on the transmit power and the error probability on each link. We use, as optimality criterion, the concept of NE [6], which is reached when each user, given the power allocation of the other users, does not get any rate increase by moving from the allocation corresponding to the equilibrium. In summary, the structure of the game is the following:

$$\mathscr{G} = \{\Omega, \{\mathscr{P}_q\}_{q\in\Omega}, \{R_q\}_{q\in\Omega}\}, \tag{3}$$

where $\Omega \triangleq \{1, 2, \ldots, Q\}$ denotes the set of the active links, $\mathscr{P}_q$ is the set of admissible power allocation strategies, across the N available sub-carriers, for the q-th player, defined as[1]

$$\mathscr{P}_q = \left\{ \mathbf{p}_q \in \mathbb{R}_+^N : \frac{1}{N} \sum_{k=0}^{N-1} p_q(k) = 1 \right\}, \tag{4}$$

and R_q is given by (1).

The full characterization of the game $\mathscr{G}$ in terms of NEs was given in [2]. Specifically, it was shown in [2] that a (pure strategy) NE always exists for $\mathscr{G}$, and that every NE can be achieved using only pure strategies. Furthermore, sufficient conditions for the uniqueness of the equilibrium were derived in [2], that enlarge all the previous ones available in the literature (such as [1, 3]).

Once proved that a Nash Equilibrium always exists and under which conditions it is unique, the problem of how to reach such an equilibrium arises. We address this issue in the next section.

3. ASYNCHRONOUS ITERATIVE WATER-FILLING

To compute the NE points of $\mathscr{G}$ in (3), we propose a simple distributed iterative water-filling procedure, called Asynchronous Iterative Water-Filling Algorithm (AIWFA), that is an instance of the totally asynchronous scheme of [5]. In AIWFA, all the users, in a *totally asynchronous* way, maximize their own rate, performing the single user water-filling solution, given the PSD of the interference generated by the other users. According to this asynchronous procedure, some users are allowed to update their strategy much more frequently than the others, and they perform these updates using probably *outdated* information on the interference caused from the others. We show in the following that, such a procedure converges to a stable NE of the game $\mathscr{G}$ in (3), under weak conditions on the multiuser interference.

In order to provide a formal description of AIWFA, we need some preliminary definitions, as we introduce next. We assume, without loss of generality, that the set of times at which one or more users update their strategies is the discrete set $T = \{0, 1, 2, \ldots\}$. Let $\mathbf{p}_q^{(n)}$ denote the power allocation of user q at the n-th iteration, and let $T_q \subseteq T$ denote the set of times n at which $\mathbf{p}_q^{(n)}$ is updated. Thus, at time $n \notin T_q$, $\mathbf{p}_q^{(n)}$ is left unchanged. At iteration n, let $\tau_r^q(n)$ denote the most recent time for which $\mathbf{p}_r$ is known to user q. Observe that $\tau_r^q(n)$ satisfies $0 \leq \tau_r^q(n) \leq n$. Hence, if user q updates its power allocation at the n-th iteration, then it waterpours the interference level caused by the following vectors

$$\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))} \triangleq \left[\mathbf{p}_1^{(\tau_1^q(n))T}, \mathbf{p}_2^{(\tau_2^q(n))T}, \ldots, \mathbf{p}_Q^{(\tau_Q^q(n))T}\right]^T. \tag{5}$$

The overall system is assumed to be totally asynchronous, i.e., if the sets $T_1, \ldots, T_Q$ are composed by infinite elements, then [5]

$$\lim_{n\to\infty} \tau_r^q(n) = \infty, \quad \forall q, r \neq q \in \Omega. \tag{6}$$

In words, given any iteration index n_1, values of the components of $\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))}$ in (5) generated prior to n_1, will not be used in the updates of $\mathbf{p}_q^{(n)}$ after a sufficiently long time n_2. This assumption guarantees that old information is eventually purged from the system.

Using the above notation, AIWFA is described in Algorithm 1.

Algorithm 1: Asynchronous Iterative Water-Filling Algorithm

Set $n = 0$ and $\mathbf{p}_q^{(0)} = \mathsf{WF}_q(\mathbf{p}_{-q})$, with any $\mathbf{p}_{-q} \geq \mathbf{0}$;

`for` $n = 1 : \mathrm{Number\_of\_iterations}$

$\forall q \in \Omega:$

$$\mathbf{p}_q^{(n+1)} = \begin{cases} \alpha \mathbf{p}_q^{(n)} + (1-\alpha)\mathsf{WF}_q\left[\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))}\right], & \text{if } n \in T_q, \\ \mathbf{p}_q^{(n)}, & \text{otherwise}; \end{cases} \tag{7}$$

`end`

In Algorithm 1, the feasible set $\mathscr{P}_q$ is defined in (4), $\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))}$ is given in (5), and $\mathsf{WF}_q[\cdot]$ denotes the water-filling operator, defined as

$$\mathsf{WF}_q[\mathbf{p}_{-q}] = \left(\mu_q \mathbf{1}_N - \mathbf{insr}_q\right)^+, \tag{8}$$

where

$$\mathbf{insr}_q \triangleq [\mathsf{insr}_q(0), \ldots, \mathsf{insr}_q(N-1)]^T, \tag{9}$$

with $\mathsf{insr}_q(k)$ given by

$$\mathsf{insr}_q(k) \triangleq \Gamma_q \frac{1 + \sum_{r\neq q} \mathsf{inr}_{rq} \left|\bar{H}_{rq}(k)\right|^2 p_r(k)}{\mathsf{snr}_q \left|\bar{H}_{qq}(k)\right|^2}. \tag{10}$$

The water-level μ_q in (8) is chosen so that the power constraint is met with equality for each transmitter. The factor $\alpha \in [0, 1)$ can be interpreted as a forgetting factor: The higher is α, the longer is the memory of the algorithm.

Since AIWFA is based on the water-filling solution (8), it can be implemented in a distributed way, where each user, to maximize its own rate, only needs to locally measure the PSD of the thermal noise plus the received multiuser interference (see (10)) and water-pour over this level. More interestingly, according to the asynchronous scheme, the users may update their strategies using a potentially outdated version of the PSD of the interference and, furthermore, some users are allowed to update their power allocation more often than others, without affecting the convergence of the algorithm. These

[1] Observe that, in the feasible strategy set of each player, we can replace, without loss of generality, the original power constraint $(1/N)\ \sum_{k=0}^{N-1} p_q(k) \leq 1$, with $(1/N)\ \sum_{k=0}^{N-1} p_q(k) = 1$, since, at the optimum, this constraint must be satisfied with equality.

features strongly relax the required constraints on the synchronization of the users' updates.

Sufficient conditions for the convergence of AIWFA are given by the following.

Theorem 1 *AIWFA, described in Algorithm 1, converges to the unique NE of the game $\mathscr{G}$ in (3), if one of the following conditions is satisfied*

$$\frac{\Gamma_q}{w_q}\sum_{r\neq q}\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\}\frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\frac{P_r}{P_q}w_r<1,\ \forall q\in\Omega, \tag{C1}$$

$$\frac{1}{w_r}\sum_{q\neq r}\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\}\frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\frac{P_r}{P_q}\Gamma_q w_q<1,\ \forall r\in\Omega, \tag{C2}$$

where $\mathbf{w}\triangleq[w_1,\ldots,w_Q]^T$ is any positive vector, and $\mathcal{D}_q$ denotes the set $\{0,\ldots,N-1\}$ possibly deprived by the subcarrier indices that user q would never use as the best response set to any strategies used by the other users, for the given set of transmit power and propagation channels. Conditions (C1)-(C2) are also sufficient for the uniqueness of the NE.

Proof. See Appendix. ■

Corollary 2 *The optimal positive vector $\mathbf{w}$ in (C1)-(C2) is given by the solution of the following geometric programming*

$$\begin{array}{ll}\underset{\mathbf{w},t}{\text{minimize}} & t\\ \text{subject to} & \sum_{r=1,r\neq q}G_{rq}t^{-1}w_q^{-1}w_r\leq 1,\quad\forall q,\\ & \mathbf{w}>\mathbf{0},\ t>0,\end{array} \tag{11}$$

where G_{rq} is defined as $G_{rq}\triangleq\Gamma_q\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{|\bar{H}_{rq}(k)|^2/|\bar{H}_{qq}(k)|^2\right\}$ $(d_{qq}^{\gamma}/d_{rq}^{\gamma})(P_r/P_q)$, if (C1) is used, or as $G_{rq}\triangleq\Gamma_r\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{|\bar{H}_{qr}(k)|^2\right.$ $\left./|\bar{H}_{rr}(k)|^2\right\}(d_{rr}^{\gamma}/d_{qr}^{\gamma})(P_q/P_r)$, if (C2) is used.

Remark 1. Note that one can always choose $\mathcal{D}_q=\{0,\ldots,N-1\}$ in (C1)-(C2). However, less stringent conditions are obtained by removing unnecessary sub-carriers, which are never used.

Remark 2. As expected, the convergence of AIWFA is ensured if the links are sufficiently far apart from each other. But, the most interesting result coming from (C1)-(C2) is that the convergence of AIWFA is robust against the worst normalized channels $|H_{rq}(k)|^2/|H_{qq}(k)|^2$; in fact, the subchannels corresponding to the highest ratios $|H_{rq}(k)|^2/|H_{qq}(k)|^2$ (and, in particular, the subchannels where $|H_{qq}(k)|^2$ is vanishing) do not necessarily affect the convergence of AIWFA, as their subcarrier indices may not belong to the set $\mathcal{D}_q$.

Remark 3. One can generalize AIWFA, including, for example, the spectral mask constraints

$$p_q(k)\leq p_q^{\max}(k),\qquad\forall q,k, \tag{12}$$

simply replacing the admissible set in (4) with the following [4]

$$\mathscr{P}_q=\left\{\mathbf{p}_q\in\mathbb{R}_+^N:1/N\ \sum_{k=0}^{N-1}p_q(k)=1,\right.\\ \left.p_q(k)\leq p_q^{\max}(k),\ \ k=0,\ldots,N-1\right\}, \tag{13}$$

and computing the water-filling solution according to the new constraints in (13) [2].

4. TWO PARTICULAR CASES

We show now that the distributed algorithms proposed in the literature to solve the rate maximization game $\mathscr{G}$ in (3), namely IWFA [1, 3] and SIWFA [4], are just special cases of the more general AIWFA. By direct product of this generalized framework, one can obtain a unified set of convergence conditions for both IWFA and SIWFA (and, more generally, for all the algorithms derived from the AIWFA). These conditions enlarge the previous ones, known in the literature [1, 3].

4.1. Sequential Iterative Water-filling

The well-known IWFA, proposed by Yu et. al. in [1], is an instance of our AIWFA. In fact, in IWFA, each player, sequentially and according to a fixed order, maximizes its own rate, given the others as interference; which corresponds to AIWFA in Algorithm 1, with a proper choice of T_q and $\tau_r^q(n)$. IWFA is described in Algorithm 2.

Algorithm 2: Sequential Iterative Water-Filling Algorithm

Set $n=0$ and $\mathbf{p}_q^{(0)}=\mathsf{WF}_q(\mathbf{p}_{-q})$, with any $\mathbf{p}_{-q}\geq\mathbf{0}$;

`for` $n=1:\text{Number\_of\_iterations}$

  `for` $q=1:Q$

$$\mathbf{p}_q^{(n+1)}=\mathsf{WF}_q\left[\mathbf{p}_1^{(n+1)},\ldots,\mathbf{p}_{q-1}^{(n+1)},\mathbf{p}_{q+1}^{(n)},\ldots,\mathbf{p}_Q^{(n)}\right] \tag{14}$$

  `end`

`end`

In Algorithm 2, $\mathsf{WF}[\cdot]$ and $\mathscr{P}_q$ are given in (8) and (4), respectively.

Sufficient conditions for the convergence of IWFA to the NE of the game $\mathscr{G}$ in (3) are given in [1] (for $Q=2$) and [3] (for $Q>2$) by

$$\Gamma_q\max_{k=0,\ldots,N-1}\left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\}\frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}<\frac{1}{Q-1},\quad\forall\, r,q\neq r\in\Omega. \tag{C3}$$

Conditions (C3) are also sufficient for the (existence [1]) uniqueness of the equilibrium.

Since IWFA in Algorithm 2 is a special case of AIWFA given in Algorithm 1, we can enlarge conditions (C3), simply invoking Theorem 1.

Theorem 3 *IWFA, described in Algorithm 2, converges to the unique NE of the game $\mathscr{G}$ in (3), if conditions (C1)-(C2) of Theorem 1 are satisfied.*

Comparing (C1)-(C2) with (C3), we infer that our conditions are larger than (C3); in fact, (C3) implies (C1)-(C2). But, the most interesting result coming from (C1)-(C2) is that, contrary to what one could infer from (C3), the convergence of IWFA is not affected by the the worst channels where every user will never transmit. This strongly relaxes conditions for the convergence, as shown, e.g., in Figure 1, where we tested (C1)-(C2) and (C3) over a set of channel impulse responses generated as vectors composed of i.i.d. complex Gaussian random variables with zero mean and unit variance. We plot in Figure 1 the probability that conditions (C1)-(C2) and (C3) are satisfied versus the ratio d_{rq}/d_{qq}, with $d_{rq}=d_{qr}$, $P_q=P_r$, $\forall r,q$, and $\Gamma_q=1$. We tested our condition considering in (C1)-(C2) the set $\mathcal{D}_q$, as obtained in [2]. We can see, from Figure 1, that the probability that (C1)-(C2) are satisfied, differently from (C3), exhibits a neat threshold behavior as it passes very rapidly from the non-convergence condition to the almost certain convergence situation, as the inter-user distance ratio d_{rq}/d_{qq} increases by a small percentage. Finally, it is worthwhile noticing that, as opposed to (C3), the inter-user distance ratios guaranteeing that (C1)-(C2) are mat are rather small, i.e. in the order of a few units. As an example, for a system with $Q=15$ links and probability of guaranteeing convergence of 0.99, conditions (C1)-(C2) only require $d_{rq}/d_{qq}\simeq 4.2$ whereas conditions (C1) require $d_{rq}/d_{qq}>45$. Furthermore, this difference increases as the number Q of links increases.

Observe that slight variations of the IWFA that fall in the unified framework of AIWFA, are still guaranteed to converge, under conditions (C1)-(C2). For example, in the Gauss-Seidel scheme of

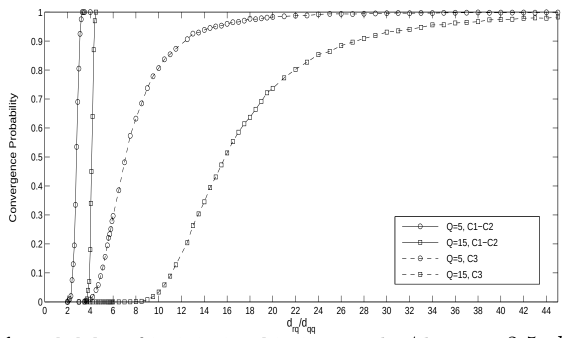

Fig. 1. Probability of (C1)-(C2) and (C3) versus d_{rq}/d_{qq}; $\gamma = 2.5$, $d_{rq} = d_{qr}$, $\forall r, q$, and $\Gamma_q = 1$.

Algorithm 2, some user may skip sometimes its update, or use an outdated version of the PSD of the interference, without affecting the convergence of the algorithm (Theorem 1).

4.2. Simultaneous Iterative Water-Filling

The Simultaneous Iterative-Water-Filling Algorithm (SIWFA) proposed in [4] can be interpreted as the synchronous Jacobi (overrelaxation) instance of the AIWFA. In fact, in SIWFA, at each iteration, all the users update their own PSD *simultaneously*, performing the single user water-filling solution (8), given the interference generated by the other users in the *previous* iteration. Setting in (7), for each q, $T_q = T$ and $\tau_r^q(n) = n - 1$, we obtain Algorithm 3.

Algorithm 3: Simultaneous Iterative-Water-Filling Algorithm

Set $n = 0$ and $\mathbf{p}_q^{(0)} = \mathsf{WF}_q\left(\mathbf{p}_{-q}\right)$, with any $\mathbf{p}_{-q} \geq \mathbf{0}$;

`for` $n = 1 : \text{Number\_of\_iterations}$

$\forall q \in \Omega$:

$$\mathbf{p}_q^{(n+1)} = \alpha \mathbf{p}_q^{(n)} + (1-\alpha)\mathsf{WF}_q\left[\mathbf{p}_1^{(n)}, \ldots, \mathbf{p}_{q-1}^{(n)}, \mathbf{p}_{q+1}^{(n)}, \ldots, \mathbf{p}_Q^{(n)}\right]; \tag{15}$$

`end`

SIWFA keeps the most appealing features of the IWFA, namely its low-complexity distributed nature. But, thanks to the Jacobi-based update, all the users are allowed to chose their optimal power allocation *simultaneously*, which makes SIWFA faster than IWFA, especially if the number of active users in the network is large [4]. Interestingly, (sufficient) conditions for the convergence of SIWFA to the unique NE of the game, are the same than those required by IWFA , as given in the following.

Theorem 4 *SIWFA, described in Algorithm 3, converges to the unique NE of the game $\mathscr{G}$ in (3), if conditions (C1)-(C2) of Theorem 1 are satisfied.*

5. CONCLUSION

In this paper, we proposed a totally asynchronous iterative distributed algorithm, called AIWFA, that solves the rate maximization game in frequency-selective multi-user interference channel. We derived sufficient conditions for the convergence of AIWFA, that represent a unified set of conditions for the convergence of all the algorithms that are special cases of AIWFA, such as IWFA and SIWFA. Surprisingly, the asynchronous mechanism in AIWFA does not impose any additional constraint on the conditions for the convergence. At the best of these knowledge, our conditions are the largest ones in the current literature.

6. APPENDIX: PROOF OF THEOREM 1

We prove that, under (C1)-(C2), AIWFA in Algorithm 1 converges to the unique NE of the game $\mathscr{G}$ in (3). To this end, we need the following intermediate results [5].

Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_Q$ be given sets, and let $\mathcal{X}$ be their Cartesian product, i.e.

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_Q. \tag{16}$$

Let $f_q : \mathcal{X} \mapsto \mathcal{X}_q$ be a given function and let $\mathbf{f} : \mathcal{X} \mapsto \mathcal{X}$ be the function defined as $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_Q(\mathbf{x})]^T$,with $\mathbf{x} \triangleq [\mathbf{x}_1^T, \ldots, \mathbf{x}_Q^T]^T \in \mathcal{X}$ and assumed to admit a fixed point $\mathbf{x}^\star = \mathbf{f}(\mathbf{x}^\star)$. Consider the following distributed asynchronous iterative algorithm to reach the fixed point $\mathbf{x}^\star$

$$\mathbf{x}_q^{(n+1)} = \begin{cases} f_q\left[\mathbf{x}_1^{(\tau_1^q(n))}, \ldots, \mathbf{x}_Q^{(\tau_Q^q(n))}\right], & \text{if } n \in T_q, \\ \mathbf{x}_q^{(n+1)}, & \text{otherwise}, \end{cases} \quad, \forall q \in \Omega; \tag{17}$$

with $0 \leq \tau_r^q(n) \leq n$ and T_q denoting the set of times n at which $\mathbf{x}_q^{(n)}$ is updated and satisfying (6).

Assume that there exists a sequence of nonempty sets $\{\mathcal{X}(n)\}$ with

$$\ldots \subset \mathcal{X}(n+1) \subset \mathcal{X}(n) \subset \ldots \subset \mathcal{X}, \tag{18}$$

satisfying the following two conditions:
1. (*Synchronous Convergence Condition*)

$$\mathbf{f}(\mathbf{x}) \in \mathcal{X}(n+1), \qquad \forall n, \text{ and } \mathbf{x} \in \mathcal{X}(n). \tag{19}$$

Furthermore, if $\{\mathbf{y}^{(n)}\}$ is a sequence such that $\mathbf{y}^{(n)} \in \mathcal{X}(n)$, for every n, then every limit point of $\{\mathbf{y}^{(n)}\}$ is a fixed point of $\mathbf{f}(\cdot)$.
2. (*Box conditions*) For every n there exist sets $\mathcal{X}_q(n) \subset \mathcal{X}_q$ such that

$$\mathcal{X}(n) = \mathcal{X}_1(n) \times \mathcal{X}_2(n) \times \ldots \times \mathcal{X}_Q(n). \tag{20}$$

Then we have the following

Theorem 5 ([5, Proposition 2.1]) *If the Synchronous Convergence condition (19) and the Box condition (20) are satisfied, and the starting point $\mathbf{x}^{(0)} \triangleq [\mathbf{x}_1^{(0)T}, \ldots, \mathbf{x}_Q^{(0)T}]^T$ of the algorithm (17) belongs to $\mathcal{X}(0)$, then every limit point of $\{\mathbf{x}^{(n)}\}$ given by (17) is a fixed point of $\mathbf{f}(\cdot)$.*

We show now that, under (C1) (or (C2)), AIWFA in Algorithm 1 satisfies Theorem 5. Given (7) and (17), we use the following identifications

$$\begin{array}{l} \mathbf{x}_q \Leftrightarrow \mathbf{p}_q, \quad \mathbf{x}_q^\star \Leftrightarrow \mathbf{p}_q^\star, \quad \mathcal{X}_q \Leftrightarrow \mathscr{P}_q, \\ f_q(\mathbf{x}) \Leftrightarrow \alpha \mathbf{x}_q + (1-\alpha)\mathsf{WF}_q\left[\mathbf{x}_{-q}\right], \\ \mathcal{X} \Leftrightarrow \mathscr{P} \triangleq \mathscr{P}_1 \times \mathscr{P}_2 \times \ldots \times \mathscr{P}_Q, \end{array} \quad \forall q \in \Omega, \tag{21}$$

with $\alpha \in [0, 1)$, and $\mathscr{P}_q$ and $\mathsf{WF}_q\left[\cdot\right]$ defined in (4) and (8), respectively.

Observe that the mapping $\mathbf{f}(\mathbf{x})$ in (21), given $\mathbf{x}$ (i.e. $\mathbf{p}$) is just the (synchronous) simultaneous water-filling update of the power allocations of the users, given the interference profile generated by $\mathbf{x}$. In [4] we proved that such a mapping satisfies the following property: if $\mathbf{p}^{(n)} \triangleq \mathbf{x} \in \mathcal{X}$, then $\mathbf{p}^{(n+1)} \triangleq \mathbf{f}(\mathbf{x})$ is such that $\mathbf{p}^{(n+1)} \in \mathcal{X}$, and

$$\mathbf{0} \leq \mathbf{e}^{(n+1)} \leq \mathbf{H}^{\max}\mathbf{e}^{(n)}, \quad n = 0, 1, \ldots. \tag{22}$$

where

$$\left[\mathbf{H}^{\max}\right]_{qr} \triangleq \begin{cases} (1-\alpha) \max\limits_{k \in \mathcal{D}_r \cap \mathcal{D}_q} \Gamma_q \dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \dfrac{d_{qq}^\gamma}{d_{rq}^\gamma} \dfrac{P_r}{P_q}, & r \neq q, \\ \gamma, & r = q, \end{cases} \tag{23}$$

and

$$\mathbf{e}^{(n)} \triangleq [e_1^{(n)}, \ldots, e_Q^{(n)}]^T, \tag{24}$$

with

$$e_q^{(n)} \triangleq \left\| \mathbf{p}_q^{(n)} - \mathbf{p}_q^{\star} \right\|_2, \quad q \in \Omega, \quad n = 0, 1, \ldots, \tag{25}$$

and $\mathbf{p}^{\star} \triangleq [\mathbf{p}_1^{\star T}, \ldots, \mathbf{p}_Q^{\star T}]^T$ denoting a NE of the game $\mathscr{G}$ in (3) (i.e. a fixed point of $\mathbf{f}(\mathbf{x})$ in (21)).

Assume first that conditions (C1) hold true. We, then show that a similar approach can be taken if conditions (C2) are satisfied. Using the weighted maximum norm

$$\|\mathbf{x}\|_{\infty}^{\mathbf{w}} = \max_i \frac{|x(i)|}{w_i}, \tag{26}$$

where $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T$ is any positive vector, we define the set $\mathcal{X}(n)$ in (18) as

$$\mathcal{X}(n) = \left\{ \mathbf{p} \in \mathscr{P} : \|\mathbf{e}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}}^n \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}} \right\} \subset \mathscr{P}, \tag{27}$$

with $\mathbf{e}$ given in (24) and $\alpha_{\mathbf{w},\mathbf{H}}$ defined as

$$\alpha_{\mathbf{w},\mathbf{H}} \triangleq \|\mathbf{H}^{\max}\|_{\infty}^{\mathbf{w}} = \max_q \frac{1}{w_q} \sum_{r=1}^{Q} w_r [H^{\max}]_{qr} < 1. \tag{28}$$

where $[H^{\max}]_{qr}$ is given in (27) and the inequality in (28) follows from the assumption that, for the given $\mathbf{w} > \mathbf{0}$, conditions (C1) hold.

Using (27), we show now that all the conditions in Theorem 5 are satisfied. Specifically:

1. *Condition* (18): From (27) it follows that

$$\mathcal{X}(n) = \left\{ \mathbf{p} \in \mathscr{P} : \|\mathbf{e}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}}^n \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}} \right\} \subset \mathscr{P}, \tag{29}$$

$$\mathcal{X}(n+1) = \left\{ \mathbf{p} \in \mathscr{P} : \|\mathbf{e}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}}^{n+1} \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}} \right\} \subset \mathscr{P}. \tag{30}$$

Since $\alpha_{\mathbf{w},\mathbf{H}} < 1$, we have

$$\alpha_{\mathbf{w},\mathbf{H}}^{n+1} \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}} < \alpha_{\mathbf{w},\mathbf{H}}^{n} \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}}, \quad \forall n = 0, 1, ...,$$

which guarantees

$$\mathcal{X}(n+1) \subset \mathcal{X}(n) \subset \mathscr{P}, \quad \forall n = 0, 1, ...,$$

as required by (18).

2. *Synchronous Convergence Condition* (19): Let $\mathbf{p}^{(n)} \in \mathcal{X}(n)$. Then, from (29), it must be

$$\|\mathbf{e}^{(n)}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}}^n \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}}, \tag{31}$$

with $\mathbf{e}^{(n)}$ given in (24). Let $\mathbf{p}^{(n+1)} = \mathbf{f}(\mathbf{p}^{(n)})$, with $\mathbf{f}(\,\cdot\,)$ defined in (21). Then, we have

$$\|\mathbf{e}^{(n+1)}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}} \|\mathbf{e}^{(n)}\|_{\infty}^{\mathbf{w}} \leq \alpha_{\mathbf{w},\mathbf{H}}^{n+1} \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}}, \tag{32}$$

where the first inequality follows from (22) and from [7]

$$\|\mathbf{H}^{\max}\mathbf{e}^{(n)}\|_{\infty}^{\mathbf{w}} \leq \|\mathbf{H}^{\max}\|_{\infty}^{\mathbf{w}} \|\mathbf{e}^{(n)}\|_{\infty}^{\mathbf{w}},$$

whereas the second inequality comes out from (31). Condition (32) guarantees $\mathbf{p}^{(n+1)} \in \mathcal{X}(n+1)$. Furthermore, since

$$\lim_{n \to \infty} \|\mathbf{e}^{(n)}\|_{\infty}^{\mathbf{w}} = 0,$$

any sequence $\{\mathbf{p}^{(n)}\}$ with $\mathbf{p}^{(n)} \in \mathcal{X}(n)$ for all n, must converge to the same $\mathbf{p}^{\star}$.

3. *Box Condition* (20): For every n, the set $\mathcal{X}(n)$ in (27) can be decomposed as $\mathcal{X}(n) = \mathcal{X}_1(n) \times \mathcal{X}_2(n) \times \ldots \times \mathcal{X}_Q(n)$, with

$$\mathcal{X}_q(n) = \left\{ \mathbf{p}_q \in \mathscr{P}_q : \frac{e_q}{w_q} \leq \alpha_{\mathbf{w},\mathbf{H}}^n \|\mathbf{e}^{(0)}\|_{\infty}^{\mathbf{w}} \right\} \subset \mathscr{P}_q, \quad \forall q \in \Omega,$$

with e_q given in (25).

Since the starting point in (7) is such that $\mathbf{p}^{(0)} \in \mathcal{X}(0)$, Theorem 5 is satisfied, which proves the convergence of AIWFA in (7) to the NE $\mathbf{p}^{\star}$ of the game $\mathscr{G}$ in (3), under conditions (C1), for any given $\mathbf{w} > \mathbf{0}$.

The optimal vector $\mathbf{w}$ is given by the following optimization problem

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} \; \max\limits_q & \sum\limits_{r=1, r\neq q} [H^{\max}]_{qr} \, w_q^{-1} w_r \\ \text{subject to} & \mathbf{w} > \mathbf{0}, \end{array}$$

which provides (11).

It is straightforward to see that the convergence of AIWFA to the NE $\mathbf{p}^{\star}$, for *any* set of initial conditions, implies that such an equilibrium is globally asymptotically stable, and thus, unique. Hence, conditions (C1) are also sufficient for the uniqueness of the NE.

Convergence of IWFA under conditions (C2), can be obtained using the following result. Since $\mathbf{H}^{\max}$ given in (23) is a nonnegative matrix, there exists a positive vector $\overline{\mathbf{w}}$ such that [5, Corollary 6.1]

$$\alpha_{\overline{\mathbf{w}},\mathbf{H}} < 1 \quad \Leftrightarrow \quad \rho\left(\mathbf{H}^{\max}\right) < 1, \tag{33}$$

where $\alpha_{\overline{\mathbf{w}},\mathbf{H}}$ is given in (28) with $\mathbf{w} = \overline{\mathbf{w}}$, and $\rho\left(\mathbf{H}^{\max}\right)$ is the spectral radius of the matrix $\mathbf{H}^{\max}$, defined as $\rho\left(\mathbf{H}^{\max}\right) = \max\{|\lambda| : \lambda \in \sigma\left(\mathbf{H}^{\max}\right)\}$ with $\sigma\left(\mathbf{H}^{\max}\right)$ denoting the spectrum of $\mathbf{H}^{\max}$ [7]. Since the convergence of AIWFA is guaranteed under (28), for any give $\mathbf{w} > \mathbf{0}$, we can choose $\mathbf{w} = \overline{\mathbf{w}}$ and obtain the alternative sufficient condition for the convergence

$$\rho\left(\mathbf{H}^{\max}\right) = \rho(\mathbf{H}^{\max\,T}) < 1. \tag{34}$$

Since [5, Proposition 6.2e]

$$\rho(\mathbf{H}^{\max\,T}) \leq \|\mathbf{H}^{\max\,T}\|_{\infty}^{\mathbf{w}}, \quad \forall \mathbf{w} > \mathbf{0}, \tag{35}$$

a sufficient condition for (34) is

$$\|\mathbf{H}^{\max\,T}\|_{\infty}^{\mathbf{w}} < 1, \tag{36}$$

which provides (C2).

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