ON EQUAL CONSTELLATION MINIMUM BER LINEAR MIMO TRANSCEIVERS

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ABSTRACT

Linear MIMO transceivers (composed of a linear precoder at the transmitter and a linear equalizer at the receiver) are a low-complexity approach to optimize the spectral efficiency and/or the reliability of the communication, when perfect channel state information is available at both sides of the link. The design of linear transceivers has been extensively studied in the literature with a variety of cost functions. In this paper we focus on the minimum BER design, and show that the common practice of fixing a priori the number of transmitted data symbols per channel use inherently limits the diversity gain of the system. Finally, we propose a minimum BER linear precoding scheme that achieves the full diversity of the MIMO channel.

Index Terms— minimum BER design, linear MIMO transceiver, analytical performance, spatial multiplexing

1. INTRODUCTION

The increasing demand of higher data rates, specifically in wireless communications, has motivated interest in the design and analysis of multiple-input multiple-output (MIMO) systems. Specifically, when channel state information is accessible simultaneously at both sides of the link, the MIMO system can be adapted to each channel realization to maximize the spectral efficiency and/or reliability of the communication. Theoretically, the optimal transmission is given by a Gaussian signaling with a waterfilling power profile over the channel eigenmodes [1]. From a more practical standpoint, however, the ideal Gaussian codes are substituted with practical constellations (such as QAM constellations) and coding schemes. To simplify the study of such a system, it is customary to divide it into an uncoded part, which transmits symbols drawn from some constellations, and a coded part that builds upon the uncoded system. Although the ultimate system performance depends on the combination of both parts, it is convenient to consider the uncoded and coded parts independently to simplify the analysis and design.

This paper focuses on the uncoded part of the system and, specifically, on the employment of linear transceivers (composed of a linear precoder at the transmitter and a linear equalizer at the receiver). The design of linear transceivers has been extensively treated in the literature according to a variety of criteria [2, 3, 4, 5]. This paper concentrates only on the design that minimizes bit error probability (BER), since it measures the ultimate performance of an uncoded digital communication system.

First, we present the minimum BER design obtained assuming that the number of data symbols per channel use (denoted by K) has been previously chosen and equal symbol constellations are employed. The minimum BER linear transceiver with fixed and equal constellations (minBERfe), derived simultaneously in [4] and [5], transmits a rotated version of the data symbols through the K strongest channel eigenmodes in a waterfilling fashion. Our analytical performance characterization shows that the diversity gain of this scheme is given by $(n_T - K + 1)(n_R - K + 1)$, where n_T is the number of transmit and n_R the number of receive antennas. This suggests that the average BER could be improved by introducing the number of active substreams in the design criterion. This extra degree of freedom is utilized in the proposed scheme by fixing the global rate but allowing the use of an adaptive symbol constellation to compensate for the change in number of active substreams. Besides, and for the sake of simplicity, the constellation is assumed to be the same in all substreams. This scheme is named minimum BER linear transceiver with fixed rate and equal constellations (minBERe), and, in contrast to the minBERfe design, does fully exploit the diversity gain $n_R n_T$ of the MIMO channel. Note that a more general setup would also adapt the individual modulations without the equal constellations constraint. However, even the minimum BER linear transceiver with fixed and unequal constellations can not be optimally obtained in closed form [6], and the proposed minBERe scheme already achieves the full diversity of the channel with low complexity.

The rest of the paper is organized as follows. Section 2 is devoted to introducing the signal model and presenting the average BER performance measure. The minimum BER linear transceiver with fixed and equal constellations design and performance analysis problem is addressed in Section 3 and with fixed rate and equal constellations in Section 4. Finally, we summarize the main contribution of the paper in the last section.

2. SYSTEM MODEL AND PERFORMANCE MEASURE

The signal model corresponding to a transmission through a general MIMO channel with n_T transmit and n_R receive antennas is

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^{n_T \times 1}$ is the transmitted vector, $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ is the channel matrix, $\mathbf{y} \in \mathbb{C}^{n_R \times 1}$ is the received vector, and $\mathbf{w} \in \mathbb{C}^{n_R \times 1}$ is a spatially white zero-mean circularly symmetric complex Gaussian noise vector normalized so that $\mathrm{E}\{\mathbf{w}\mathbf{w}^\dagger\} = \mathbf{I}_{n_R}$.

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The channel matrix $\mathbf{H}$ contains the complex path gains $[\mathbf{H}]_{ij}$ between every transmit and receive antenna pair. We adopt an uncorrelated Rayleigh (Ricean) flat-fading channel model and, consequently these coefficients are independent complex Gaussian random variables with zero (non-zero) mean and unit variance.

Suppose that the MIMO communication system is equipped with a linear transceiver, then the transmitted vector is given by

$$\mathbf{x} = \mathbf{B}_K \mathbf{s}_K. \tag{2}$$

where $\mathbf{B}_K \in \mathbb{C}^{n_T \times K}$ is the transmit matrix (precoder) and the data vector $\mathbf{s}_K \in \mathbb{C}^{K \times 1}$ gathers the $K \leq \min\{n_T, n_R\}$ data symbols to be transmitted (zero mean, with unit energy and uncorrelated, i.e. $\mathrm{E}\{\mathbf{s}_K \mathbf{s}_K^\dagger\} = \mathbf{I}_K$). We consider a fixed-rate data transmission and, hence, each data symbol $s_{k,K}$ is drawn from a fixed M_K-dimensional constellation such that the total transmission rate $R = K \log_2 M_K$[1] is fixed for all channel realizations and all possible values of K. The transmitted power is constrained such that

$$\mathrm{E}\left\{\|\mathbf{x}\|^2\right\} = \mathrm{Tr}\{\mathbf{B}_K \mathbf{B}_K^\dagger\} \leq \mathsf{snr} \tag{3}$$

where snr is the average SNR per receive antenna. The estimated data vector at the receiver is

$$\hat{\mathbf{s}}_K = \mathbf{A}_K^\dagger \mathbf{y} = \mathbf{A}_K^\dagger \left(\mathbf{H}\mathbf{B}_K \mathbf{s}_K + \mathbf{w}\right). \tag{4}$$

where $\mathbf{A}_K^\dagger \in \mathbb{C}^{K \times n_R}$ is the receive matrix (equalizer).

The ultimate performance of MIMO linear transceivers is measured by the BER averaged over the K data symbols to be transmitted:

$$\mathrm{BER}_K(\mathsf{snr}) = \frac{1}{K}\sum_{k=1}^{K} \mathrm{BER}_{k,K}(\rho_{k,K}) \tag{5}$$

where $\rho_{k,K}$ is the instantaneous SNR and $\mathrm{BER}_{k,K}(\rho_{k,K})$ is the instantaneous BER of the k^{th} substream (out of K active) given by

$$\mathrm{BER}_{k,K}(\rho_{k,K}) = \frac{\alpha_K}{\log_2 M_K} \mathcal{Q}\left(\sqrt{\beta_K \rho_{k,K}}\right) \tag{6}$$

and α_K and β_K are parameters of the M_K-dimensional constellation (see e.g. [7]).

3. MINIMUM BER LINEAR TRANSCEIVER WITH FIXED AND EQUAL CONSTELLATIONS

3.1. Linear Transceiver Design

The general problem of designing the optimal linear MIMO transceiver (when fixing K beforehand) is formulated in [4] as the minimization of a certain cost function of mean-square errors (MSEs), since the BER can be easily related to the MSE. Specifically, [4] shows that the optimum receive matrix $\mathbf{A}_K$, for a given transmit matrix $\mathbf{B}_K$, is given by the Wiener filter solution:

$$\mathbf{A}_K = \left(\mathbf{H}\mathbf{B}_K \mathbf{B}_K^\dagger \mathbf{H}^\dagger + \mathbf{I}_{n_R}\right)^{-1} \mathbf{H}\mathbf{B}_K \tag{7}$$

independently of the design cost function. Under the minimum BER design criterion, the precoder matrix $\mathbf{B}_K$ is obtained as

$$\mathbf{B}_K = \arg\min_{\mathbf{B}_K} \mathrm{BER}_K(\mathsf{snr}) \tag{8}$$

[1] Note that K and $\log_2 M_K$ have to be integers.

subject to the power constraint in (3). The optimum transmit matrix $\mathbf{B}_K$ is given by [4, 5]

$$\mathbf{B}_K = \mathbf{U}_K \sqrt{\mathbf{P}_K} \mathbf{Q}_K \tag{9}$$

where $\mathbf{U}_K \in \mathbb{C}^{n_T \times K}$ has as columns the eigenvectors of $\mathbf{H}^\dagger\mathbf{H}$ corresponding to the K largest nonzero eigenvalues $\lambda_1 \geq \cdots \geq \lambda_K$, $\mathbf{Q}_K \in \mathbb{C}^{K \times K}$ is a unitary matrix such that $\left(\mathbf{I}_K + \mathbf{B}_K^\dagger \mathbf{H}^\dagger \mathbf{H} \mathbf{B}_K\right)^{-1}$ has identical diagonal elements (see [4] for details), and $\mathbf{P}_K \in \mathbb{C}^{K \times K}$ is a diagonal matrix with diagonal entries equal to

$$p_{k,K} = \left(\mu \lambda_k^{-1/2} - \lambda_k^{-1}\right)^+ \qquad k = 1, 2, \ldots, K \tag{10}$$

where μ is chosen to satisfy the power constraint in (3) with equality.

3.2. Analytical Performance

Given the optimum transmit matrix in (9) and the optimum receive matrix in (7), the communication process is diagonalized up to a specific rotation that forces all data symbols to have the same MSE [4]

$$\mathsf{mse}_K \triangleq \mathsf{mse}_{k,K} = \frac{1}{K}\sum_{i=1}^{K} \left(p_{i,K}\lambda_i + 1\right)^{-1} \tag{11}$$

and, hence, the same instantaneous SNR [4]

$$\rho_K \triangleq \rho_{k,K} = \mathsf{mse}_K^{-1} - 1 = \left(\frac{1}{K}\sum_{i=1}^{K} \left(p_{i,K}\lambda_i + 1\right)^{-1}\right)^{-1} - 1. \tag{12}$$

Thus, the minBERfe design transmits a rotated version of the K data symbols through the K strongest channel eigenmodes, so that all data symbols experience the same BER performance. The instantaneous BER averaged over the K data symbols defined in (5) is then given by

$$\mathrm{BER}_K(\mathsf{snr}) = \frac{\alpha_K}{\log_2 M_K} \mathcal{Q}\left(\sqrt{\beta_K \rho_K}\right). \tag{13}$$

Now, taking into account different channel realizations, the average BER is obtained as

$$\begin{aligned}\overline{\mathrm{BER}}_K(\mathsf{snr}) &= \mathrm{E}\left\{\mathrm{BER}_K(\mathsf{snr})\right\} \\ &= \frac{\alpha_K}{\log_2 M_K}\int_0^\infty \mathcal{Q}\left(\sqrt{\beta_K \rho}\right) f_{\rho_K}(\rho) d\rho \end{aligned} \tag{14}$$

where $f_{\rho_K}(\rho)$ is the pdf of the instantaneous SNR, ρ_K, given in (12). Under the Rayleigh (Ricean) fading assumption, ρ_K is a function of the K strongest eigenvalues of the Wishart [8] distributed channel matrix $\mathbf{H}^\dagger\mathbf{H}$. Since tractable close-form expressions for the marginal pdfs of the ordered eigenvalues have not been derived, the average BER in (14) cannot be analytically computed. A convenient method to find a simple performance measure is to allow a certain degree of approximation and shift the focus from exact performance to large SNR performance as done in [7, 9].

Based on the previous work by Wang and Giannakis in [10], the average BER versus SNR curves of the channel eigenmodes have been characterized in terms of two key parameters (the array gain and the diversity gain) for a Rayleigh fading channel in [7] and for a Ricean fading channel in [9]. The diversity gain represents the slope of the BER curve at high SNR and the array gain (also known as coding gain) determines the horizontal shift of the BER curve. The following theorem extends the previous procedure to characterize the average BER performance of the minBERfe design in which the number of active substreams is fixed beforehand.

Theorem 1 *The average BER attained by the minimum BER linear transceiver with fixed and equal constellations (assuming K data symbols per channel use) in an uncorrelated Rayleigh or Ricean flat-fading MIMO channel is*

$$\overline{\mathrm{BER}}_K(\mathsf{snr}) = (G_a \cdot \mathsf{snr})^{-G_d} + o\left(\mathsf{snr}^{-G_d}\right) \tag{15}$$

where the diversity gain is given by

$$G_d = (n_T - K + 1)(n_R - K + 1) \tag{16}$$

and the array gain can be bounded as

$$G_{a,\mathrm{lb}} < G_a < G_{a,\mathrm{ub}} \tag{17}$$

where $G_{a,\mathrm{lb}}$ is the global array gain when using a diagonal scheme with a uniform power allocation and is given by [7]

$$G_{a,\mathrm{lb}} = \frac{\beta_K}{K}\left(\frac{\alpha_K}{K \log_2 M_K} \frac{a_K 2^{d_K}}{\sqrt{\pi}(d_K+1)}\right)^{-1/(d_K+1)} \tag{18}$$

and $G_{a,\mathrm{ub}}$ is defined as

$$G_{a,\mathrm{ub}} = \beta_K \left(\frac{\alpha_K}{\log_2 M_K} \frac{a_K I(d_K, \beta_K(K-1))}{\sqrt{2\pi}(d_K+1)}\right)^{-1/(d_K+1)} \tag{19}$$

where $I(d, \beta\phi)$ is[2]

$$I(d, \beta\phi) = \int_{\sqrt{\beta\phi}}^{\infty} e^{-\frac{x^2}{2}} \left(x^2 - \beta\phi\right)^{(d+1)} dx. \tag{20}$$

The parameters a_K and d_K model the pdf of the K^{th} ordered channel eigenvalue as

$$f_{\lambda_K}(\lambda_K) = a_K \lambda_K^{d_K} + o(\lambda_K^{d_K}) \tag{21}$$

and their expressions are given in [7] for the Rayleigh fading and in [9] for the Ricean fading channel.

Proof: *See [11].*

It is important to note that the diversity gain given in Theorem 1 coincides with the diversity gain achieved when transmitting K symbols in parallel with equal power over the K strongest channel eigenmodes [7, 9]. Hence, Theorem 1 shows that the minBERfe design does not provide any diversity advantage with respect to diagonal schemes with uniform power allocation policies but only a higher array gain. This statement is confirmed by Fig. 1, where we show the average BER performance of the minBERfe scheme (solid lines) and of the diagonal scheme with uniform power allocation (dashed lines) in a Rayleigh flat-fading channel. We also provide the parameterized upper and lower bounds (dash-doted lines) derived from Theorem 1. It turns out that the proposed array gain upper bound is in fact very tight and approximates perfectly the high SNR performance.

Intuitively, the performance of this scheme is limited by the inherent performance of K^{th} channel eigenmode, because the design cost function in (5) is evaluated for the K data symbols to be transmitted, regardless whether $p_{K,K} = 0$ or not. This reveals that the average BER can be improved by introducing K into the design criterion, as analyzed in the following section.

[2] A closed-form expression for this integral does not exist for a general value of the parameter d; however, it can be easily evaluated for the most common values of d (integers).

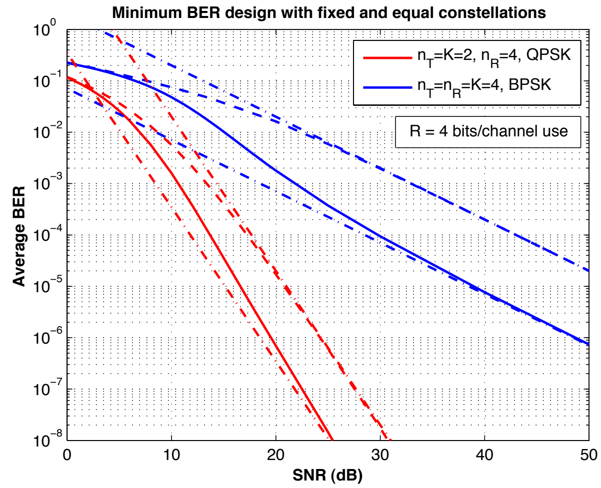

Fig. 1. Simulated average BER and parameterized average BER bounds (dash-dotted) of the minBERfe linear transceiver (solid) and the diagonal linear transceiver with uniform power allocation (dashed).

4. MINIMUM BER LINEAR TRANSCEIVER WITH FIXED RATE AND EQUAL CONSTELLATIONS

4.1. Linear Transceiver Design

The precoding process is here slightly different from classical linear precoding, where the number of data symbols to be transmitted per channel use K is fixed. In the proposed scheme, K and the M_K-dimensional constellations are adapted to the instantaneous channel conditions by allowing K to vary between 1 and $n = \min\{n_T, n_R\}$ keeping the total transmission rate $R = K \log_2 M_K$ fixed. Usually, only a subset $\mathcal{K}$ of all n possible values of K is supported, since the number of bits per symbol R/K has to be an integer [12].

The linear precoder $\mathbf{B}_K$ and K are designed to minimize the BER averaged over the data symbols to be transmitted for all supported values of K:

$$\{K, \mathbf{B}_K\} \quad = \quad \arg \min_{K,\, \mathbf{B}_K} \mathrm{BER}_K(\mathsf{snr}) \tag{22}$$

where $\mathrm{BER}_K(\mathsf{snr})$ is defined in (5), $K \in \mathcal{K}$, and $\mathbf{B}_K$ has to satisfy the power-constraint in (3). The optimum linear precoder $\mathbf{B}_K$ for a given K has been presented and analyzed in Section 3. Then, the optimum K should be selected as

$$K = \arg \min_{K \in \mathcal{K}} \frac{\alpha_K}{\log_2 M_K} \mathcal{Q}\left(\sqrt{\beta_K \rho_K}\right) \tag{23}$$

or, neglecting the contribution of $\alpha_K / \log_2 M_K$ (since it is not in the argument of the $\mathcal{Q}$-function), as

$$K = \arg \max_{K \in \mathcal{K}} \beta_K \rho_K \tag{24}$$

where ρ_K is given in (12).

4.2. Analytical Performance

The average BER performance of the minBERe design is even more dificult to derive than the minBERfe scheme analyzed in Theorem 1. Hence, we have only characterized the diversity gain as presented in the following theorem.

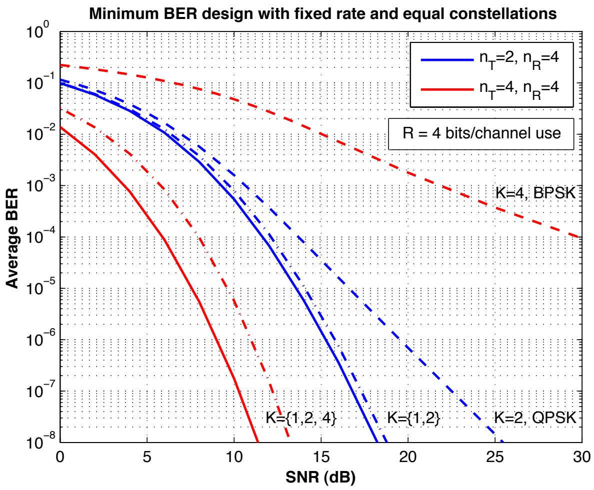

Fig. 2. Simulated average BER of the minBERe linear transceiver (solid), the minBERfe linear transceiver (dashed), and the multimode precoder (dash-doted).

Theorem 2 *The diversity gain attained by the minimum BER linear transceiver with fixed rate and equal constellations in an uncorrelated Rayleigh or Ricean flat-fading MIMO channel is bounded by*

$$n_T n_R \geq G_d \geq (n_T - K_{\min} + 1)(n_R - K_{\min} + 1). \qquad (25)$$

where $K_{\min}$ denotes the minimum value of K in $\mathcal{K}$.

Proof: The average BER of the scheme under analysis, denoted by $\overline{\mathrm{BER}}(\mathsf{snr})$, can be upper-bounded using Jensen's inequality as

$$\overline{\mathrm{BER}}(\mathsf{snr}) = \mathrm{E}\left\{\min_{K\in\mathcal{K}} \mathrm{BER}_K(\mathsf{snr})\right\} \qquad (26)$$

$$\leq \min_{K\in\mathcal{K}} \overline{\mathrm{BER}}_K(\mathsf{snr}) \leq \overline{\mathrm{BER}}_{K_{\min}}(\mathsf{snr}). \qquad (27)$$

where $\overline{\mathrm{BER}}_{K_{\min}}(\mathsf{snr})$ denotes the average BER of the minBERfe design when $K = K_{\min}$. Then, using Theorem 1, the diversity gain of the minimum BER linear transceiver with fixed rate and equal constellations is lower-bounded by $(n_T - K_{\min}+1)(n_R - K_{\min}+1)$. The upper bound corresponds with the full diversity offered by the channel. □

Theorem 2 shows that the minimum BER linear transceiver with fixed rate and equal constellations effectively exploits the maximum diversity offered by the MIMO channel whenever $K = 1$ is contained in $\mathcal{K}$. In Fig. 2 we plot the average BER performance of the minBERe scheme (solid lines) in a Rayleigh flat-fading channel. In addition, we have included the minBERfe design (dashed lines) and a suboptimum scheme (dash-dotted lines) that also adapts K jointly with the precoder and achieves full diversity. This technique is called multimode precoding and was proposed in [12] in the context of limited feedback linear precoding. As expected, the proposed design outperforms the classical minBERfe linear transceiver and the suboptimum scheme of [12].

5. CONCLUSIONS

This paper has characterized the average BER performance of the minimum BER linear transceiver with fixed and equal constellations in a Rayleigh/Ricean flat-fading channel. It turns out that this classical minimum BER design has a diversity order limited by that of the worst eigenmode used, which can be far from the full diversity provided by the channel. This shows that fixing a priori the number of independent data streams to be transmitted, a very common assumption in the linear transceiver design literature, inherently limits the average BER performance of the system. Based on this observation, we have considered the minimum BER linear transceiver with fixed rate and equal constellations and shown that it achieves the full diversity of the channel thanks to optimizing the number of substreams jointly with the linear precoder.

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